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Laminar heat transfer of non-Newtonian fluid in the entrance region of a circular conduit.

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LAMINAR HEAT TRANSFER OF NON-NEWTONIAN FLUID
IN THE ENTRANCE REGION OF A CIRCULAR CONDUIT

BY

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ABSTRACT

An analytical study of laminar heat transfer of non-Newtonian fluid in the entrance region of a circular conduit was carried out in this thesis. An asymptotic solution is obtained for the entrance heat transfer problem, with velocity and temperature profiles developing simultaneously. The non-Newtonian fluid is assumed to be of the "power-law" model and its physical properties are assumed to be constant. The initial velocity and temperature profiles of the fluid prior to its entry are considered to be flat, and the walls of the conduit are maintained at uniform but different temperatures from that of the fluid. Numerical values of local and average Nusselt numbers as functions of Prandtl number and dimensionless longitudinal distances have been evaluated and presented in graphical forms for both Newtonian and non-Newtonian fluids.

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besides being of academic interest, is also of practical importance.

Fluids are classified into the two main categories of Newtonian or non-Newtonian according to their behavior at constant temperature under imposed shearing forces. Newtonian fluids are those which exhibit a direct proportionality between shear stress and shear rate in the laminar-flow region. By definition the term "non-Newtonian" includes all materials which do not obey the direct proportionality between shear stress and shear rate.

Metzner (11), following the classical method of classifying non-Newtonian fluids divides non-Newtonian systems into three broad categories:

(1) Fluids with properties independent of time or duration of shear (time-independent non-Newtonian).

(2) More complex fluids for which the relationship between shear stress and shear rate depends upon the duration of shear (time-dependent non-Newtonian).

(3) Those systems which have many characteristics of a solid and primarily that of elastic recovery from the deformations which occur upon flow.

Time-independent non-Newtonians are further divided into three distinct groups:

(a) Bingham plastics

(b) pseudoplastics

(c) dilatant fluids

From the fluid-flow curves presented in Fig. A, the flow behavior of each type can be compared with the Newtonian fluid.

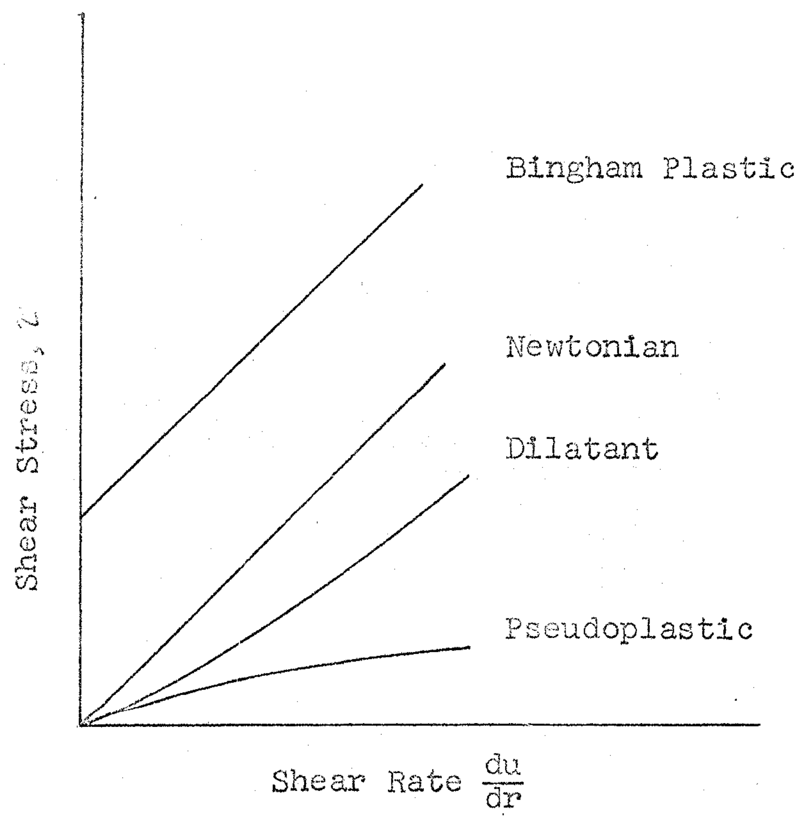

Fig. A Fluid-Flow Curves

Examples of fluids which have been stated to approximate Bingham-plastic behavior are drilling muds, suspensions of chalk, grains, rock and sewage sludge. The Bingham model which expresses the rheological relation between the shear stress (τ_{rz}) and the shear rate ($-\frac{du}{dr}$) may be written for the one-dimensional case as:

$$\tau_{rz} = -\mu_o \frac{du}{dr} \pm \tau_o \qquad |\tau_{rz}| > \tau_o \qquad (1\text{-}1)$$

$$\frac{du}{dr} = 0 \qquad |\tau_{rz}| < \tau_o \qquad (1\text{-}2)$$

where τ_o is the yield stress.

Pseudoplastic fluids comprise the classification into which the majority of industrially important non-Newtonian fluids fall. Examples of fluids which exhibit this type of behavior include polymeric solutions or melts, as rubbers, cellulose acetate, and Napalm; suspensions such as paints, mayonaise, paper pulp, and detergent slurries; and even dilute suspensions of inert, unsolvated solids. The rheological behavior can be described most simply by the "power-law" model, for the one-dimensional case as:

$$\tau_{rz} = M\left(-\frac{du}{dr}\right)^n \qquad (1\text{-}3)$$

Other models that can be used are:

(1) Eyring model

(2) Ellis model

(3) Reiner-Philippoff model

Dilatant fluids display a rheological behavior opposite to that of pseudoplastics in that the apparent viscosity increases with increasing shear rate. The usual examples of dilatant behavior are starch, potassium silicate, and gum arabic in water.

Time-dependent non-Newtonians may be divided into two groups, depending on whether the shear stress increases or decreases with time of shear at a constant shearing rate. The former are termed rheopectic, the latter thixotropic fluids.

In this investigation, consideration is given only to pseudoplastic fluids whose rheological behavior can be described by the "power-law" model.

Laminar flow of various non-Newtonian models have been investigated (3,11,16), both theoretically and experimentally. The velocity development in the entrance region has been studied and reported by Bogue (4) and recently by Collins and Schowalter (5). Heat transfer studies and in particular the extension of the Graetz-Nusselt problem to non-Newtonian flow, which we are here primarily interested in, have been carried out by Tien (14) and Lyche and Bird (10).

Those existing solutions referred to are generally based on the assumption of a fully established velocity profile at the point in the tube where heating begins. Entrance region heat transfer studies, when both velocity and temperature profiles are developing, have been made only recently.

Exact analysis of this kind of problem is difficult due to the complexities of the governing differential equations. A numerical solution was given by Kays (7) for the case of Newtonian fluid in a circular conduit. In Kays' work, the energy equation was simplified by omitting the term $V_r \frac{\partial T}{\partial r}$ and the numerical results were obtained by difference approximation using the developing velocity profile information given by Langhaar (9). Kays' work was later extended by Goldberg (6), to enlarge the range of the Prandtl number of the fluid. Sparrow (13) solved this problem for flat ducts using the momentum integral method. Yau and Tien (15), also by use of the momentum integral method solved this problem for non-Newtonian fluid in the case of flat ducts.

In this investigation an asymptotic solution will be used to solve the entrance effect for a circular conduit with non-Newtonian fluids involved. The advantage of this approach is that a closed-form expression can be obtained. This will be discussed in detail in following sections.

CHAPTER II

STATEMENT OF PROBLEM AND METHOD OF SOLUTION

The problem to be studied here is the solution of laminar convective heat transfer in the entrance region of a circular conduit where velocity and temperature profiles are developing simultaneously.

Fig. 1 gives the schematic description of the problem under investigation. A constant-property fluid, assumed to have a uniform velocity profile and uniform temperature, enters a round tube. The wall of the tube is maintained at a constant temperature T_w where $T_w \neq T_o$. For convenience in this mathematical study, cylindrical coordinates are used with r denoting the radial coordinate while z denotes the axial coordinate. Similarly, V_r and V_z denote the velocity components in the radial and longitudinal directions respectively.

Besides those assumptions already mentioned, the following are also employed:

(1) The flow is two dimensional.

(2) The flow is steady.

(3) The dissipative heat due to friction is negligible.

(4) The fluid is incompressible and has constant physical properties.

(5) The longitudinal conduction of heat is insignificant.

(6) There is a non-slip condition at the wall. Under this condition, there is a thin layer of fluid adjacent to the wall in which the velocity is zero at the wall, but approaches to very near main flow velocity at a distance δ from the wall. This fluid layer is called the velocity boundary layer of thickness δ, and outside this layer, potential flow occurs in the core and the core velocity profile is flat. The velocity boundary layer is assumed to be of zero thickness at the inlet point. It increases in thickness with distance from the inlet point until it reaches the centre line of the duct where the two boundary layers from both walls merge. It is assumed that the viscous effect is confined within the boundary layer, and outside the region the forces due to friction are small and may be neglected.

(7) There is a transfer of heat between the fluid and the wall because of the temperature difference. The major part of this transfer takes place in a thin layer of fluid adjacent to the wall. Within the layer temperature varies from T_w at the wall to T_o of the flow in the core. In an exactly analogous manner to the velocity boundary layer, this thin layer is called the thermal boundary layer. Outside the thermal boundary layer the fluid is not materially affected by the heat transfer and the temperature remains the same as that of the fluid before entering. This layer will grow from zero thickness at the inlet point to a value equal to R at the centre line where the two identical thermal boundary layers from both walls meet. It is further assumed that the effect of longitudinal conduction of heat in the fluid is insignificant.

(8) The usual boundary layer simplifications can be applied to the equations of motion and energy.

With the above assumptions taken into consideration, the following equations of continuity, motion and energy may be written as follows:

$$\frac{1}{r}\frac{\partial}{\partial r}(r V_r) + \frac{\partial V_z}{\partial z} = 0 \tag{2-1}$$

$$V_r \frac{\partial V_z}{\partial r} + V_z \frac{\partial V_z}{\partial z} = -\frac{1}{\rho}\frac{\partial P}{\partial z} - \frac{1}{\rho r}\frac{\partial}{\partial r}(r \tau_{rz}) \tag{2-2}$$

$$V_r \frac{\partial T}{\partial r} + V_z \frac{\partial V_z}{\partial z} = \alpha \left[\frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial T}{\partial r}\right)\right] \tag{2-3}$$

The shear stress term τ_{rz} of Equation (2-2) can be expressed in terms of the rate of strain if the rheological model of the fluid is specified.

There are several methods which may be employed in the solution of the stated problem. Exact analysis is not feasible mainly because of the complexities of the governing differential equations. This leaves approximate methods of solution as the main field of study.

One approach is to further simplify the basic equations given above by omitting certain terms and obtaining numerical results by difference analysis of developing velocity profile information. This is the method employed by Kays (7).

Another approach involves the use of momentum integral method for the approximate solution of the velocity and temperature profiles. Although the use of the momentum integral method is quite simple, it suffers from a basic weakness, namely that there is no criterion available to assess the accuracy of the results.

The method to be employed in this investigation is to obtain an asymptotic solution of the entrance region heat transfer problem for a circular conduit. From Equation (2 -2) through a physical consideration of the relative thickness of the thermal and velocity boundary layers for high Prandtl number fluids, a simplified expression for velocity distribution can be used in the energy equation and enables the solution of this problem.

CHAPTER III

ASYMPTOTIC SOLUTION

Considering the physical properties of the fluid to be constant the equation of motion can be solved independent of the energy equation. The solution gives V_z as a function of both space coordinates (r and z), which was first obtained by Langhaar.

In the present work, solution of Equation (2-3) is to be obtained by simplifying the energy equation from physical argument. For fluids with large Prandtl numbers it is known that the thickness of the thermal boundary layer is much thinner in comparison with that of velocity boundary layer. For this reason, it is possible to approximate the part of the velocity profile within the thermal boundary layer by the following expression:

$$V_z = -\left(\frac{\partial V_z}{\partial r}\right)_R (R - r) \tag{3-1}$$

This is analogous to the approximation of Graetz-Nusselt solution by Leveque's solution. First writing Equations (2-1), (2-3) and (3-1) into dimensionless forms, we have

$$\frac{1}{1-y^+}\frac{\partial}{\partial y}\left[(1-y^+)V_y^+\right] - \frac{\partial V_z^+}{\partial Z^+} = 0 \qquad (3\text{-}2)$$

$$V_y^+\frac{\partial T^+}{\partial y^+} - V_z^+\frac{\partial T^+}{\partial Z^+} = \frac{1}{Pr\,Re}\left[\frac{\partial^2 T^+}{\partial y^{+2}} - \frac{1}{1-y^+}\frac{\partial T^+}{\partial y^+}\right] \qquad (3\text{-}3)$$

$$V_z^+ = \beta(z^+)\,y^+ \qquad (3\text{-}4)$$

where

$$y^+ = 1 - r^+ = 1 - \frac{r}{R} \qquad (3\text{-}5)$$

$$Z^+ = \frac{Z}{R} \qquad (3\text{-}6)$$

$$V^+ = \frac{V}{\bar{U}} \qquad (3\text{-}7)$$

$$T^+ = \frac{T - T_o}{T_w - T_o} \qquad (3\text{-}8)$$

$$\beta(z^+) = -\left(\frac{\partial V_z^+}{\partial r^+}\right)_{r^+=1} = \left(\frac{\partial V_z^+}{\partial y^+}\right)_o \qquad (3\text{-}9)$$

$Pr = \dfrac{C_p\mu}{k}$ for Newtonian fluid (3-10a)

$\quad = \dfrac{2C_p\rho R\bar{U}}{k\,Re}$ for non-Newtonian fluid (3-10b)

$Re = \dfrac{R\bar{U}\rho}{\mu}$ for Newtonian fluid (3-11a)

$\quad = \dfrac{\rho\,2^n R^n U^{2-n}}{M}$ for non-Newtonian fluid (3-11b) and boundary conditions become

$$T^+ = 1 \qquad\qquad y^+ = 0 \tag{3-12}$$

$$T^+ = 0 \qquad\qquad z^+ = 0 \tag{3-13}$$

Combining Equations (3-2) and (3-3), we have

$$V_r^+ = \beta'(z^+)\,\frac{3y^{+2} - y^{+3}}{6(1-y^+)} \tag{3-14}$$

Substituting Equations (3-4) and (3-14) into (3-3), we have

$$\beta y^+ \frac{\partial T^+}{\partial z^+} - \beta' \frac{3y^{+2} - y^{+3}}{6(1-y^+)} \frac{\partial T^+}{\partial y^+} - \frac{1}{PrRe} + \frac{1}{1-y^+}\frac{\partial T^+}{\partial y^+} = \frac{1}{PrRe}\frac{\partial^2 T^+}{\partial y^{+2}} \tag{3-15}$$

Since one is primarily concerned with the temperature distribution within the thermal boundary layer which is very thin in the entrance region for fluids with high Prandtl numbers, we can simplify Equation (3-15) by approximating

$$1 - y^+ \approx 1$$

$$3y^{+2} - y^{+3} \approx 3y^{+2}$$

Also let

$$\phi = \frac{\beta}{\sqrt{Re}} \tag{3-16}$$

$$u = y^+\sqrt{Re} \tag{3-17}$$

Equation (3-15) becomes

$$\phi u \frac{\partial T^+}{\partial z^+} - \left(\phi' \frac{u^2}{2} - \frac{1}{Pr\sqrt{Re}} \right) \frac{\partial T^+}{\partial u} = \frac{1}{Pr}\left(\frac{\partial^2 T^+}{\partial u^2} \right) \qquad (3\text{-}18)$$

and

$$T^+ = 1 \qquad u = 0$$

$$T^+ = 0 \qquad z^+ = 0$$

Assume that the expression of T^+ can be written as

$$T^+ = T_0^+ + \frac{T_1^+}{Pr} + \frac{T_2^+}{(Pr)^2} + \cdots\cdots \qquad (3\text{-}19)$$

Substituting Equation (3-19) into (3-18), we have

$$\phi u \left(\frac{\partial T_0^+}{\partial z^+} + \frac{1}{Pr}\frac{\partial T_1^+}{\partial z^+} + \cdots \right) - \left(\phi' \frac{u^2}{2} + \frac{1}{Pr\sqrt{Re}} \right)\left(\frac{\partial T_0^+}{\partial u} + \frac{1}{Pr}\frac{\partial T_1^+}{\partial u} + \cdots \right)$$
$$= \left(\frac{1}{Pr} \right)\left(\frac{\partial^2 T_0^+}{\partial u^2} + \frac{1}{Pr}\frac{\partial^2 T_1^+}{\partial u^2} + \cdots \right) \qquad (3\text{-}20)$$

Equation (3-20) will be satisfied if we require that

$$\phi u \frac{\partial T_0^+}{\partial z^+} - \phi' \frac{u^2}{2}\frac{\partial T_0^+}{\partial u} = \frac{1}{Pr}\frac{\partial^2 T_0^+}{\partial u^2} \qquad (3\text{-}21a)$$

$$\phi u \frac{\partial T_1^+}{\partial z^+} - \phi \frac{u^2}{2}\frac{\partial T_1^+}{\partial u} = \frac{1}{Pr}\frac{\partial^2 T_1^+}{\partial u^2} - \frac{1}{\sqrt{Re}}\frac{\partial T_0^+}{\partial u} \qquad (3\text{-}21b)$$

$$\phi u \frac{\partial T_2^+}{\partial z^+} - \phi \frac{u^2}{2}\frac{\partial T_2^+}{\partial u} = \frac{1}{Pr}\frac{\partial^2 T_2^+}{\partial u^2} - \frac{1}{\sqrt{Re}}\frac{\partial T_1^+}{\partial u} \qquad (3\text{-}21c)$$

or, in general

$$\phi u \frac{\partial T_i^+}{\partial z^+} - \phi \frac{u^2}{2}\frac{\partial T_i^+}{\partial u} = \frac{1}{Pr}\frac{\partial^2 T_i}{\partial u^2} - \frac{1}{\sqrt{Re}}\frac{\partial T_{i-1}}{\partial u} \qquad (3\text{-}21d)$$

for $i \geqslant 1$

The boundary conditions are

$$T_o^+ = 1 \qquad u^+ = 0 \tag{3-22a}$$

$$T_i^+ = 0 \qquad u^+ = 0 \;, \quad i \geq 1 \tag{3-22b}$$

and

$$T_i^+ = 0 \qquad Z^+ = 0 \;, \quad i \geq 0 \tag{3-23}$$

Equation (3-21a) together with the boundary condition given by (3-22a) and (3-23) are identical with those of heat transfer over a flat plate for fluids with high Prandtl numbers and its solution has been given before (1), (2) and is given as

$$T_o^+ = 1 - \frac{\int_0^{\eta} e^{-\eta^3} d\eta}{\int_0^{\infty} e^{-\eta^3} d\eta} \tag{3-24}$$

when

$$\eta = u \left(\frac{Pr}{9}\right)^{1/3} \frac{\phi^{1/2}}{\left[\int_0^{Z^+} \phi^{\frac{1}{2}} dZ^+\right]^{1/3}} \tag{3-25}$$

and

$$\left(\frac{\partial T_o^+}{\partial \eta}\right)_o = - \frac{1}{\int_0^{\infty} e^{-\eta^3} d\eta} \tag{3-26}$$

$$\left(\frac{\partial T_o^+}{\partial u}\right) = (-1)\left(\frac{Pr}{9}\right)^{\frac{1}{3}} \frac{\phi^{1/2}}{\left[\int_0^{z^+} \phi^{\frac{1}{2}}\, dz^+\right]^{1/3}} \frac{e^{-\eta^3}}{\int_0^\infty e^{-\eta^3} d\eta} \tag{3-27}$$

Combining Equations (3-27) and (3-21b) the result is:

$$\phi u \frac{\partial T_1^+}{\partial z^+} - \phi' \frac{u^2}{2} \frac{\partial T_1^+}{\partial u}$$

$$= \frac{1}{Pr} \frac{\partial^2 T_1^+}{\partial u^2} + \frac{1}{\sqrt{Re}} \left(\frac{Pr}{9}\right)^{1/3} \frac{\phi^{1/2}}{\left[\int_0^{z^+} \phi^{1/2}\, dz^+\right]^{1/3}} \tag{3-28}$$

Because of the additional term on the right hand side of Equation (3-28), a similarity transformation such as that given by Equation (3-25) will not be applicable. However, by introducing two new independent variables η and ζ with η defined as before and ζ is given as

$$\zeta = z^+ \tag{3-29}$$

Expressing T_1^+ as well as its derivatives in terms of η and ζ and substituting into Equation (3-28) after rearrangement, we have

$$\frac{\partial^2 T_1^+}{\partial \eta^2} + 3\eta^2 \frac{\partial T_1^+}{\partial \eta}$$

$$= \frac{9\int_0^{z^+} \phi^{1/2} dz^+}{\phi^{1/2}} \eta \frac{\partial T_1^+}{\partial \zeta} - \frac{3^{2/3} P_r^{2/3}}{\sqrt{Re}\ \phi^{1/2}} \left[\int_0^{z^+} \phi^{1/2} dz^+\right]^{1/2} \frac{e^{-\eta^3}}{\int_0^{\infty} e^{-\eta^3} d\eta}$$

(3-30)

One may argue that for small values of η, the first term on the right hand side can be dropped. Also for large values of η, the term $\partial T_0^+/\partial u$ can be omitted (see Equation (3-27)). Equation (3-21b) would become the same with Equation (3-21a) and $\partial T_1^+/\partial \zeta$ would be identical to zero. Therefore one can write for approximation, that

$$\frac{\partial^2 T_1}{\partial \eta^2} - 3\eta \frac{\partial T_1}{\partial \eta} = -\frac{3^{2/3} P_r^{2/3}}{\sqrt{Re}\ \phi^{1/2}} \left[\int_0^{z^+} \phi^{1/2} dz^+\right]^{1/2} \frac{e^{-\eta^3}}{\int_0^{\infty} e^{-\eta^3} d\eta} \qquad (3\text{-}31)$$

The boundary conditions as stated before are

$$T_1^+ = 0 \qquad \eta = 0$$
$$T_1^+ \rightarrow 0 \qquad \eta \rightarrow \infty$$

Solution of Equation (3-31) can be found to be

$$T_1 = -\frac{3^{2/3} P_r^{2/3}}{\sqrt{Re}\,\phi^{1/2}}\left[\int_0^{z^+} \phi^{1/2} dz^+\right]^{1/3} \frac{\int_0^\infty \eta e^{-\eta^3} d\eta}{\int_0^\infty e^{-\eta^3} d\eta} + \left(\frac{\partial T_1^+}{\partial \eta}\right)_0 \int_0^\eta e^{-\eta^3} d\eta \tag{3-32}$$

and

$$\left(\frac{\partial T_1^+}{\partial \eta}\right)_0 = \frac{3^{2/3} P_r^{2/3}}{\sqrt{Re}\,\phi^{1/2}}\left[\int_0^{z^+} \phi^{1/2} dz^+\right]^{1/3} \frac{\int_0^\infty \eta e^{-\eta^3} d\eta}{\int_0^\infty e^{-\eta^3} d\eta} \tag{3-33}$$

By the same procedure as before, one can solve for T_2^+ and evaluate $\left(\partial T_2^+/\partial\eta\right)_0$. The results can be summarized as

$$\left(\frac{\partial T_0^+}{\partial \eta}\right)_0 = \frac{-1}{\int_0^\infty e^{-\eta^3} d\eta} \tag{3-34a}$$

$$\left(\frac{\partial T_1^+}{\partial \eta}\right)_0 = \frac{P_r\left[\int_0^{z^+} \phi^{1/2} dz^+\right]^{1/3}}{\left(\frac{P_r}{9}\right)^{1/3} Re^{1/2}\,\phi^{1/2}} \frac{1}{\int_0^\infty e^{-\eta^3} d\eta} \frac{\int_0^\infty \eta e^{-\eta^3} d\eta}{\int_0^\infty e^{-\eta^3} d\eta}$$

$$= \frac{P_r}{X}\cdot\frac{A_1}{\int_0^\infty e^{-\eta^3} d\eta} = \frac{P_r}{X}\cdot\frac{B_1}{\int_0^\infty e^{-\eta^3} d\eta} \tag{3-34b}$$

and in general

$$\left(\frac{\partial T_K^+}{\partial \eta}\right)_0 = \left(\frac{Pr}{X}\right)^K \frac{(-1)^{K+1}}{\int_0^\infty e^{-\eta^3} d\eta} B_K \tag{3-34c}$$

Where

$$X = \left(\frac{Pr}{9}\right)^{1/3} \sqrt{Re} \; \frac{\phi^{1/2}}{\left[\int_0^{z^+} \phi^{1/2} dz^+\right]^{1/3}} \tag{3-35}$$

$$B_0 = 1 \tag{3-36a}$$

$$B_1 = B_0 A_1 \tag{3-36b}$$

$$B_2 = B_1 A_1 - B_0 A_2 / 2! \tag{3-36c}$$

$$B_3 = B_2 A_1 - \frac{B_1 A_2}{2!} + \frac{B_0 A_3}{2!\,3!} \tag{3-36d}$$

and

$$B_K = B_{K-1} A_1 - \frac{B_{K-2} A_2}{2!} + \frac{B_{K-3} A_3}{2!\,3!} - \cdots (-1)^{K+1} \frac{B_0 A_K}{2!\,3!\cdots K!} \tag{3-36e}$$

$$A_1 = \frac{\int_0^\infty \eta e^{-\eta^3} d\eta}{\int_0^\infty e^{-\eta^3} d\eta} = \frac{\Gamma(2/3)}{\Gamma(1/3)} \tag{3-37a}$$

$$A_2 = \frac{\int_0^\infty 2 e^{-\eta^3} d\eta}{\int_0^\infty e^{-\eta^3} d\eta} = \frac{\Gamma(1)}{\Gamma(1/3)} \tag{3-37b}$$

$$A_3 = \frac{\int_0^\infty 3 e^{-\eta^3} d\eta}{\int_0^\infty e^{-\eta^3} d\eta} = \frac{\Gamma(4/3)}{\Gamma(1/3)} \tag{3-37c}$$

and in general

$$A_K = \frac{\int_0^\infty \eta^K e^{-\eta^3} d\eta}{\int_0^\infty e^{-\eta^3} d\eta} = \frac{\Gamma(\frac{K+1}{3})}{\Gamma(1/3)} \tag{3-37d}$$

Combining Equations (3-36) and (3-37), the first four values of B_k's are found to be

$$B_0 = 1$$
$$B_1 = 0.5044$$
$$B_2 = 0.0677$$
$$B_3 = -0.0461$$

The absolute magnitude of B_k's seems to be decreasing. For the numerical evaluation to be discussed later, only three terms will be required to give the desired accuracy.

The local Nusselt number is defined as

$$Nu_z = \frac{q \cdot R}{k(T_0 - T_w)} = \frac{\left(\frac{\partial T}{\partial r}\right)_R R}{(T_0 - T_w)} = -\left(\frac{\partial T^+}{\partial y^+}\right)_0 \tag{3-38}$$

From Equation (3-17), (3-25) and (3-35), one has

$$\left(\frac{\partial T^+}{\partial y^+}\right)_0 = \left(\frac{\partial T^+}{\partial \eta}\right)_0 \sqrt{Re}\left(\frac{Pr}{9}\right)^{1/3} \frac{\phi^{1/2}}{\left[\int_0^{z^+} \phi^{1/2} dz^+\right]^{1/3}}$$

$$= X \cdot \left(\frac{\partial T^+}{\partial \eta}\right)_0 \tag{3-39}$$

The expression for $\left(\frac{\partial T^+}{\partial \eta}\right)_0$ can be obtained by combining Equations (3-19) and (3-34), giving

$$Nu_z = X\left(\frac{\partial T^+}{\partial \eta}\right)_0 = \frac{X}{\int_0^\infty e^{-\eta^3} d\eta}\left(B_0 - \frac{B_1}{X} + \frac{B_2}{X^2} - \frac{B_3}{X^3} + \cdots\right) \tag{3-40}$$

with B_k's given by Equation (3-36).

The average Nusselt number is found to be

$$Nu_{avg} = \frac{\int_0^{z^+} Nu_z \, dz^+}{\int_0^{z^+} dz^+} \tag{3-41}$$

Equation (3-40) and (3-41) provide the basis for computing the Nusselt number. These expressions are given in terms of parameter X defined by Equation (3-35). The magnitude of X is dependent on the physical properties of the fluid (Prandtl number), the flow condition (Reynolds number) and the shearing stress function

$$\frac{\phi^{1/2}}{\left[\int_0^{z^+} \phi^{1/2} dz^+\right]^{1/3}}$$

CHAPTER IV

ENTRANCE HEAT TRANSFER SOLUTION FOR NEWTONIAN FLUID

Although the purpose of this investigation is to study the entrance heat transfer problem for non-Newtonian fluid, the asymptotic solution obtained in the previous chapter is a general solution and applicable to both Newtonian and non-Newtonian fluid. For completeness, the case of Newtonian fluid will be studied first. Furthermore this enables a comparison between the asymptotic solution and the numerical results obtained earlier by Goldberg (6).

From Equations (3-40) and (3-41), the Nusselt number is found to be an unique function of the parameter X which in turn, is defined by Equation (3-35). The task now is to evaluate X in terms of the developing velocity data which was obtained by Langhaar (9). From Equations (3-35) and (3-16) one has

$$X = \sqrt{Re}\left(\frac{Pr}{q}\right)^{1/3} \frac{\phi^{1/2}}{\left[\int_0^{z^+} \phi^{1/2}\, dz^+\right]^{1/2}}$$

$$= \sqrt{Re}\left(\frac{Pr}{9}\right)^{1/3} \frac{\left(\frac{\beta}{\sqrt{Re}}\right)^{1/2}}{\left[\int_0^{z^+}\left(\frac{\beta}{\sqrt{Re}}\right)^{1/2} dz^+\right]^{1/3}}$$

$$= \left(\frac{Pr}{9}\right)^{1/3}(Re)^{1/3} \frac{\beta^{1/2}}{\left[\int_0^{z^+}\beta^{1/2}\, dz^+\right]^{1/3}} = \left(\frac{Pr}{9}\right)^{1/3} \frac{\beta^{1/2}}{\left[\int_0^{\frac{z}{R \cdot Re}}\beta^{1/2}\, d\left(\frac{z}{R \cdot Re}\right)\right]^{1/3}} \tag{4-1}$$

where β is given as

$$\beta = -\left(\frac{\partial V_z^+}{\partial r^+}\right)_1 \tag{4-2}$$

The developing velocity profile obtained by Langhaar (9) expressed in terms of present nomenclature is given as

$$V_z^+ = \left[I_0(\gamma) - I_1(\gamma r^+)\right] \Big/ I_2(\gamma) \tag{4-3}$$

where γ is a parameter dependent upon the dimensionless longitudinal distance $\frac{z}{R.Re}$. Differentiating Equation (4-3) and substituting into Equation (3-9) gives

$$\beta = \frac{\gamma I_1(\gamma)}{I_2(\gamma)} \tag{4-4}$$

Langhaar's results give β as a function of the longitudinal distance, $\frac{z}{R.Re}$. Combining this information with Equation (4-1), one can evaluate X in terms of the two parameters, $\frac{z}{R.Re}$ and Pr. The relationship of $\frac{X}{(Pr/9)^{1/3}}$ vs. $\frac{z}{R.Re}$ is given in Table 3, as well as Fig. 3-a. With the relationship of $\frac{X}{(Pr/9)^{1/3}}$ vs. $\frac{z}{R.Re}$ established, one can evaluate the local Nusselt number from Equation (3-40) and similarly for average Nusselt number, from Equation (3-41). These results are given in Tables 5 and 7, and Figs. 4-a and 5-a.

In the evaluation of $\frac{X}{(Pr/9)^{1/3}}$ vs. $\frac{z}{R.Re}$ from Equation (3-35) some uncertainty was encountered for small values of $\frac{z}{R.Re}$ since $\beta \to \infty$ as $\frac{z}{R.Re} \to 0$ and graphical integration for the integral

$$\int_0^{\frac{z}{R\cdot Re}} \beta^{1/2}\, d\left(\frac{z}{R\cdot Re}\right) \qquad (4\text{-}5)$$

becomes unreliable. This difficulty was overcome by the following consideration:

The flow behavior in the immediate region after its entry point should be approximated by the flat plate problem. The flat plate solution is known to be (12),

$$\beta = \left(\frac{\partial V_z^+}{\partial y^+}\right)_0 = \frac{0.332}{\left(\frac{z}{R\cdot Re}\right)^{1/2}} \qquad (4\text{-}6)$$

Fig. 2 gives the plot of β vs. $\frac{z}{R.Re}$ based on Equations (4-2) and (4-3). It can be seen that Langhaar's results appear to be approaching asymptotically to those from Equation (4-3) with the exception of the first point. It is suspected that there might be some error in its evaluation and consequently this point was omitted and Langhaar's results were extrapolated to be asymptotically approaching the flat plate solution. From the result of Fig. 2, values of X were evaluated as functions of Pr and $\frac{z}{R.Re}$. This is shown in Fig. 3. In all cases, values of $\frac{z}{R.Re}$ range from 5×10^{-6} to 1×10^{-1} where the flow is almost fully developed.

CHAPTER V

ENTRANCE HEAT TRANSFER FOR NON-NEWTONIAN FLUID

Development of Velocity Profile

As shown in Chapter III, the heat transfer information as expressed in terms of Nusselt number can be obtained directly from Equations (3-40) and (3-41) provided the developing velocity information is available. For a power-law non-Newtonian fluid flowing in a circular conduit, the developing velocity problem was first solved by Bogue (4) using momentum integral method with a second degree polynomial for velocity approximation. Although Bogue's work could be used for evaluating heat transfer information, it is felt that more accurate information on velocity development can be obtained if more terms are used in the velocity expression. Consequently the developing velocity profile is re-calculated. This is shown as follows.

For a constant-property fluid in the entrance region of a circular conduit, the equation of motion and continuity equation are given as

$$V_r \frac{\partial V_z}{\partial r} + V_z \frac{\partial V_z}{\partial z} = -\frac{1}{\rho}\frac{\partial P}{\partial z} - \frac{1}{\rho r}\frac{\partial}{\partial r}(r\tau_{rz}) \tag{5-1}$$

$$\frac{\partial V_z}{\partial z} + \frac{V_r}{r} + \frac{\partial V_r}{\partial r} = 0 \tag{5-2}$$

outside boundary layer, one can write

$$U\frac{dU}{dz} = -\frac{1}{\rho}\frac{dP}{dz} \tag{5-3}$$

Combining Equation (5-1) and (5-3) gives

$$V_r \frac{\partial V_z}{\partial r} + V_z \frac{\partial V_z}{\partial z} = U\frac{dU}{dz} - \frac{1}{\rho r}\frac{\partial}{\partial r}(r\tau_{rz}) \tag{5-4}$$

multiplying Equation (5-4) by r and integrating from $r = R$ to $r = R - \delta$, one has

$$\int_R^{R-\delta} r V_r \frac{\partial V_z}{\partial r}\,dr + \int_R^{R-\delta} V_z \frac{\partial V_z}{\partial z}\,r\,dr$$

$$= \int_R^{R-\delta} U\frac{dU}{dz}\,r\,dr - \frac{1}{\rho}\left(r\tau_{rz}\Big|_R^{R-\delta}\right)$$

$$= \int_R^{R-\delta} U\frac{dU}{dz}\,r\,dr + \frac{1}{\rho}R\,\tau_{rz}\Big|_R \tag{5-5}$$

also from the equation of continuity, we have

$$\frac{\partial}{\partial r}(r V_r) = -r\frac{\partial V_z}{\partial z}$$

$$\int_R^{R-\delta} \frac{\partial}{\partial r}(r V_r)\, dr = -\int_R^{R-\delta} r\frac{\partial V_z}{\partial z}\, dr$$

$$(r V_r)\Big|_{R-\delta} = -\int_R^{R-\delta} r\frac{\partial V_z}{\partial z}\, dr \qquad (5\text{-}6)$$

Boundary Conditions:

$V_z = 0$	at	$r = R$	(5-7)
$V_z = U$	at	$r = R - \delta$	(5-8)
$\frac{\partial V_z}{\partial r} = 0$	at	$r = R - \delta$	(5-9)

The first term of Equation (5-5) can be written as

$$\int_R^{R-\delta} r V_r \frac{\partial V_z}{\partial r}\, dr = r V_r V_z\Big|_R^{R-\delta} - \int_R^{R-\delta} V_z \frac{\partial}{\partial r}(r V_r)\, dr$$

$$= U(r V_r)_{R-\delta} + \int_R^{R-\delta} V_z\, r\frac{\partial V_z}{\partial z}\, dr$$

$$= -U\int_R^{R-\delta} \frac{\partial V_z}{\partial z}\, r\, dr + \int_R^{R-\delta} V_z\, r\frac{\partial V_z}{\partial z}\, dr$$

so Equation (5-5) can be written as

$$- U \int_R^{R-\delta} \frac{\partial V_z}{\partial z} r\,dr + \int_R^{R-\delta} \frac{\partial}{\partial z} (V_z^2)\, r\,dr$$

$$= \int_R^{R-\delta} U \frac{dU}{dz} r\,dr + \frac{1}{\rho} R\, \tau_{rz}\Big|_R$$

or

$$\frac{dU}{dz} \int_R^{R-\delta} (U - V_z)\, r\,dr + \frac{dU}{dz} \int_R^{R-\delta} V_z\, r\,dr + U \int_R^{R-\delta} \frac{\partial V_z}{\partial z} r\,dr$$

$$- \int_R^{R-\delta} \frac{\partial}{\partial z} (V_z^2)\, r\,dr$$

$$= - \frac{1}{\rho} R\, \tau_{rz}\Big|_R$$

or

$$\frac{dU}{dz} \int_R^{R-\delta} (U - V_z)\, r\,dr + \int_R^{R-\delta} \frac{\partial}{\partial z} \left[V_z (U - V_z) \right] r\,dr$$

$$= - \frac{1}{\rho} R\, \tau_{rz}\Big|_R \qquad (5\text{-}10)$$

It is permissible to interchange the integral and the differential operator since the term $V_z(U - V_z)$ vanishes for both upper and lower limits. Also for fluids obeying the power-law model, the shear stress tensor τ and the rate of strain tensor Δ are related to be

$$\tau = -\left\{ M \left| \sqrt{\frac{1}{2}\, \Delta : \Delta} \right|^{n-1} \right\} \Delta$$

using boundary layer simplification for cylindrical coordinates, one can show that

$$\tau_{rz} = -M \left| \frac{\partial V_z}{\partial r} \right|^{n-1} \frac{\partial V_z}{\partial r}$$

or Equation (5-10) becomes

$$\frac{dU}{dz}\int_{R}^{R-\delta} (U - V_z)\, r\, dr + \frac{d}{dz}\int_{R}^{R-\delta} V_z (U - V_z)\, r\, dr$$

$$= \frac{M}{\rho} R \left| \left(\frac{\partial V_z}{\partial r}\right)_R \right|^{n-1} \left(\frac{\partial V_z}{\partial r}\right)_R \qquad (5\text{-}11)$$

A polynomial of the fourth degree is assumed for the velocity distribution inside the boundary layer. The form of this distribution is written as

$$\frac{V_z}{U} = C_1\left(\frac{y}{\delta}\right) + C_2\left(\frac{y}{\delta}\right)^2 + C_3\left(\frac{y}{\delta}\right)^3 + C_4\left(\frac{y}{\delta}\right)^4 \tag{5-12}$$

$$\text{where } y = R - r \tag{5-13}$$

The coefficients C_1, C_2, C_3 and C_4 can be determined by compatibility conditions which are given by Equations (5-7) to (5-9). Equation (5-12) is automatically satisfied by Equation (5-7). Two more conditions are required to determine uniquely the values of C_1, C_2, C_3 and C_4.

For a fully developed flow, the velocity profile is given as

$$\frac{V_z}{U} = 1 - \left[1 - \left(\frac{y}{R}\right)\right]^{\frac{n+1}{n}} = 1 - \left(\frac{r}{R}\right)^{\frac{n+1}{n}} \tag{5-14}$$

The assumed velocity expression, Equation (5-12) should take the form of Equation (5-14) when $\delta = R$. Thus the coefficients C_i can be determined for n = 1, 1/2 and 1/3. Appendix A.1 gives a sample calculation for the determination of the coefficients C_i by matching Equation (5-12) with Equation (5-14).

For values of n lower than 1/3, Equation (5-12) cannot be matched with Equation (5-14) since it is obvious that one cannot make a polynomial expression of lower degree to be identical with one of higher degree. Determination of the coefficients C_i are to be made with the following requirements:

(1) The total volumetric flow rate across the fully developed section as calculated by the approximate velocity expression should be the same as that calculated by the exact one [Equation (5-14)]. This gives

$$\int_0^R \left[C_1\left(1-\frac{r}{R}\right)+C_2\left(1-\frac{r}{R}\right)^2+C_3\left(1-\frac{r}{R}\right)^3+C_4\left(1-\frac{r}{R}\right)^4\right]2\pi r\,dr$$

$$=\int_0^R\left[1-\left(\frac{r}{R}\right)^{\frac{n+1}{n}}\right] r\,dr \qquad (5\text{-}15)$$

(2) The kinetic energy of the fluid crossing the fully developed section as determined by the approximate and exact expressions should be equal, i.e.:

$$\int_0^R \left[C_1\left(1-\frac{r}{R}\right) + C_2\left(1-\frac{r}{R}\right)^2 + C_3\left(1-\frac{r}{R}\right)^3 + C_4\left(1-\frac{r}{R}\right)^4\right]^2 r\,dr$$

$$= \int_0^R \left[1-\left(\frac{r}{R}\right)^{\frac{n+1}{n}}\right] r\,dr \qquad (5\text{-}16)$$

Hence Equations (5 - 8), (5 - 9), (5-15) and (5-16) complete the necessary conditions for the determination of the coefficients in the assumed velocity distribution Equation (5-12).

From Equation (5 - 8) $\quad C_1 + C_2 + C_3 + C_4 = 1 \qquad (5\text{-}17)$

From Equation (5 - 9) $\quad C_1 + 2C_2 + 3C_3 + 4C_4 = 0 \qquad (5\text{-}18)$

From Equation (5-15) $\quad \frac{C_1}{3} + \frac{C_2}{6} + \frac{C_3}{10} + \frac{C_4}{15} \qquad (5\text{-}19)$

$$= 1 - \frac{2n}{1 + 3n}$$

From Equation (5-16)

$$C_1^2(\tfrac{1}{12}) + C_2^2(\tfrac{1}{30}) + C_3^2(\tfrac{1}{56}) + C_4^2(\tfrac{1}{90}) + C_1C_2(\tfrac{1}{10}) + C_1C_3(\tfrac{1}{15}) + C_1C_4(\tfrac{1}{21}) + C_2C_3(\tfrac{1}{21}) + C_2C_4(\tfrac{1}{28}) + C_3C_4(\tfrac{1}{36}) = \frac{1}{2} - \frac{2n}{3n+1} + \frac{n}{4n+2} \tag{5-20}$$

Appendix A.2 gives a sample calculation for the determination of C_i by Equations (5-17) to (5-20). It should be noted that the above equations fail to determine C_i for $n = 0.10655$, because the fourth power polynomial approximation of the velocity distribution is no longer sufficient to describe the flow behavior.

Numerical values of C_i for various values of the flow behavior index (n) are given in Table 1. A comparison between the assumed expression for $\frac{V_z}{U}$ at fully developed condition and that of the exact, for various values of the flow behavior index (n) is shown in Fig. 7.

It has been shown that the fully developed velocity distribution is given as

$$\frac{U}{\bar{U}} = \frac{1+3n}{1+n}$$

where $\bar{U}$ is the average velocity of the fluid.

Therefore

$$\frac{V_z}{\bar{U}} = \frac{V_z}{U}\left(\frac{1+3n}{1+n}\right) \tag{5-21}$$

The results of $\frac{V_z}{\overline{U}}$ are plotted in Fig. 8.

When the approximate velocity expression Equation (5-12) is substituted into the momentum integral equation Equation (5-11), and after integration,

$$\frac{dU}{dz}\left(K_1 U R \delta + K_2 U \delta^2 \right) + \frac{d}{dz}\left(K_3 U^2 R \delta - K_4 U^2 \delta^2 \right)$$

$$= \frac{M R C_1^n U^n}{\rho\, \delta^n} \tag{5-22}$$

where

$$K_1 = 1 - \frac{C_1}{2} - \frac{C_2}{3} - \frac{C_3}{4} - \frac{C_4}{5} \tag{5-23}$$

$$K_2 = -\frac{1}{2} + \frac{C_1}{3} + \frac{C_2}{4} + \frac{C_3}{5} + \frac{C_4}{6} \tag{5-24}$$

$$K_3 = \frac{C_1}{2} + \frac{C_2}{3} + \frac{C_3}{4} + \frac{C_4}{5} - \frac{C_1^{\ 2}}{3} - \frac{C_2^{\ 2}}{5} - \frac{C_3^{\ 2}}{7} - \frac{C_4^{\ 2}}{9}$$
$$- \frac{C_1 C_2}{2} - \frac{2C_1 C_3}{5} - \frac{C_1 C_4}{3} - \frac{C_2 C_3}{3} - \frac{2C_2 C_4}{7} - \frac{C_3 C_4}{4} \tag{5-25}$$

$$K_4 = \frac{C_1}{3} + \frac{C_2}{4} + \frac{C_3}{5} + \frac{C_4}{6} - \frac{C_1^{\ 2}}{4} - \frac{C_2^{\ 2}}{6} - \frac{C_3^{\ 2}}{8}$$
$$- \frac{C_4^{\ 2}}{10} - \frac{2C_1 C_2}{5} - \frac{C_1 C_3}{3} - \frac{2C_1 C_4}{7} - \frac{2C_2 C_3}{7}$$
$$- \frac{C_2 C_4}{4} - \frac{2C_3 C_4}{9} \tag{5-26}$$

The non-Newtonian fluid enters the tube at a uniform velocity (U_∞). As the fluid moves along the tube, the fluid within the velocity boundary layer near the wall is retarded due to the viscous effect, while the fluid in the core will be accelerated from the initial uniform velocity to a final velocity which is attained when the two boundary layers meet at the centre line. This continuity requirement can be expressed as

$$\bar{U}R^2 = \int_{R-\delta}^{R} 2V_z\, r\, dr + U(R-\delta)^2 \tag{5-27}$$

or

$$\frac{1}{2}R^2(\bar{U} - U) = \int_{R-\delta}^{R} (U - V_z)\, r\, dr$$

$$= -U\int_0^{\delta} \left(1 - \frac{V_z}{U}\right)(R-y)\, dy$$

$$= -K_1 U R \delta - K_2 U \delta^2 \tag{5-28}$$

Let

$$\frac{U}{\bar{U}} = U^* \tag{5-29}$$

then

$$K_2\delta^2 - K_1 R\delta = \frac{1}{2}R^2\left(1 - \frac{1}{U^*}\right) \tag{5-30}$$

or

$$\delta = \frac{R}{2} \cdot \frac{K_1}{K_2} \left[-1 - \sqrt{1 - \frac{2K_2}{K_1^2}\left(1 - \frac{1}{U^*}\right)} \right] \tag{5-31}$$

Let

$$\frac{\delta}{R} = \delta^* \tag{5-32}$$

then

$$\delta^* = \frac{K_1}{2K_2} \left[-1 - \sqrt{1 - \frac{2K_2}{K_1^2}\left(1 - \frac{1}{U^*}\right)} \right] \tag{5-33}$$

Substitute Equations (5-30) and (5-31) into Equation (5-22) and carry out the operation:

$$\frac{dU}{dz} \frac{1}{2} R^2 (U - \overline{U}) - \frac{d}{dz} \left\{ K_3 U^2 \frac{R^2 K_1}{2K_2} \left[-1 + \sqrt{1 + \frac{2K_2}{K_1^2}\left(1 - \frac{1}{U^*}\right)} \right] \right.$$

$$\left. - K_4 U^2 \frac{R^2}{4} \frac{K_1^2}{K_2^2} \left[1 - 2\sqrt{1 - \frac{2K_2}{K_1^2}\left(1 - \frac{1}{U^*}\right)} \right] - 1 + \frac{2K_2}{K_1^2}\left(1 - \frac{1}{U^*}\right) \right\}$$

$$= \frac{MRC_1^{\,n} U^n}{\rho \delta^n} \tag{5-34}$$

or

$$\frac{K_1^n \left[1-\sqrt{1+\frac{2K_2}{K_1^2}\left(1-\frac{1}{U^*}\right)}\right]^n}{C_1^n (K_2)^n 2^{2n}} \left\{ U^* \left(\frac{1}{2} - \frac{K_1K_3}{K_2} - \frac{K_4K_1^2}{K_2^2} - \frac{K_4}{K_2}\right)\right.$$

$$+ \frac{K_4}{2K_2} - \frac{1}{2} + \frac{K_1}{K_2} U^* \sqrt{1+\frac{2K_2}{K_1^2}\left(1-\frac{1}{U^*}\right)} \left[K_3 + \frac{K_4K_1}{K_2}\right]$$

$$\left. + \frac{1}{2\sqrt{1+\frac{2K_2}{K_1^2}\left(1-\frac{1}{U^*}\right)}} \left[\frac{K_3}{K_1} + \frac{K_4}{K_2}\right]\right\} \frac{dU^*}{U^{*n}}$$

$$= d\left(\frac{z}{\frac{R\rho D^n \bar{U}^{2-n}}{M}}\right) \tag{5-35}$$

Let

$$Re_d = \frac{\rho D^n \bar{U}^{2-n}}{M} \tag{5-36}$$

$$z^* = \frac{z}{R \cdot Re_d} \tag{5-37}$$

then Equation (5-35) becomes

$$\left\{ \frac{-K_1\left[1-\sqrt{1+\frac{2K_2}{K_1^2}\left(1-\frac{1}{U^*}\right)}\right]}{4\,U^*\,C_1\,K_2} \right\}^n \left\{ U^*\left(\frac{1}{2}-\frac{K_1K_3}{K_2}-\frac{K_4K_1^2}{K_2^2}-\frac{K_4}{K_2}\right)\right.$$

$$+\frac{K_4}{2K_2}+\frac{1}{2}-\frac{K_1U^*}{K_2}\sqrt{1+\frac{2K_2}{K_1^2}\left(1-\frac{1}{U^*}\right)}\left[K_3+\frac{K_4K_1}{K_2}\right]$$

$$\left.+\frac{1}{2\sqrt{1+\frac{2K_2}{K_1^2}\left(1-\frac{1}{U^*}\right)}}\left[\frac{K_3}{K_1}+\frac{K_4}{K_2}\right]\right\} dU^*$$

$$= dz^* \qquad (5\text{-}38)$$

Equation (5-38) was integrated numerically over the interval $Z^* = 0$ and $U^* = 1$ to $U^* =$ fully developed value, i.e. $\delta^* = 1$. Appendix B gives a sample calculation. Table 2 gives the results of U^* and δ^* as a function of Z^* for $n = 1/4$, $1/2$ and $3/4$; these are also presented graphically in Fig. 9 and Fig. 10.

In the neighbourhood of the entrance section, $Z^* = 0$ and $U^* = 1$, Equation (5-38) is integrated to be

$$\frac{1}{n+2}\,\frac{K_3 K_1}{2K_2}\left(\frac{K_1}{4C_1 K_2}\right)^{n}\left[1-\sqrt{1+\frac{2K_2}{K_1^2}(U^*-1)}\right]$$

$$+\left[\sqrt{1+\frac{2K_2}{K_1^2}(U^*-1)}-\frac{1}{1+n}\right]$$

$$= Z^* \tag{5-39}$$

At the fully developed section, $\delta^* = 1$, Equation (5-33) becomes

$$U^* = \frac{1}{1 - 2K_2 - 2K_1} \tag{5-40}$$

Heat Transfer Results

Just as in the case where Newtonian fluid was involved the Nusselt number may be obtained as

$$Nu_z = \frac{X}{\int_0^\infty e^{-\eta^3} d\eta}\left(B_0 - \frac{B_1}{X} + \frac{B_2}{X^2} + \frac{B_3}{X^3} - \cdots\right) \tag{5-41}$$

with B_K's given by Equation (3-36) and X defined as in Equation (4 - 1). However for non-Newtonian solution the developing velocity profile found in the first part of this chapter is used. Recalling that β by definition is

$$\beta = \left(\frac{\partial V_z}{\partial y}\right)_0$$

By transforming the assumed velocity profile into dimensionless quantities and differentiating it is found that

$$\beta = U^{*} C_1 / \delta^{*} \tag{5-42}$$

Thus β is determined for the range of values computed and tabulated in Appendix F and may be substituted into Equation (5-41) to determine the local Nusselt number by numerical integration. The first value of β to be evaluated by Equation (5-42) using the results in Table 2 corresponds to a value of 5.929×10^{-5} for $\frac{z}{R.Re}$. Since β is found to approach infinity as $\frac{z}{R.Re}$ tends to zero then once again a problem for the computation of the integral

$$\int_0^{\frac{z}{R Re}} \beta^{1/2} \, d\left(\frac{Z}{R \cdot Re}\right)$$ is encountered.

In order to find values of β for the immediate regions after entry point of the fluid, one may make the approximation that $U^{*} = 1$ for very small Z^{*}. The differential equation (5-38) may then be reduced to the following form: (See Appendix C for actual manipulation.)

$$P (U^{*} - 1)^{n} \, dU^{*} = dz^{*} \tag{5-43}$$

where

$$P = \left(\frac{K_2}{K_1^2}\right)^n \left\{ \left(\frac{K_1}{4C_1K_2}\right) \left[\left(\frac{1}{2} - \frac{K_1K_3}{K_2} - \frac{K_4K_1^2}{K_2^2} - \frac{K_4}{K_2}\right) + \right.\right.$$

$$\left.\left. \left(\frac{K_4}{2K_2} - \frac{1}{2}\right) + \frac{K_1}{K_2}\left(K_3 + \frac{K_4K_1}{K_2}\right) + \frac{1}{2}\left(\frac{K_3}{K_1} + \frac{K_4}{K_2}\right)\right]\right\} \tag{5-44}$$

Integration of Equation (5-43) gives

$$(U^* - 1)^{n+1} = \frac{n+1}{P} Z^* \tag{5-45}$$

For very small values of Z^* the dimensionless quantity δ^* is reduced to

$$\frac{U^* - 1}{2} \tag{5-46}$$

By combining Equation (5-45) with (5-46) and substituting into (5-42), and expression for β for small values of Z^* is

$$\beta = 2C_1 \left(\frac{P}{n+1}\right)^{\frac{1}{n+1}} Z^{*^{-\frac{1}{n+1}}} \tag{5-47}$$

Thus the integral $$\int_0^{\frac{z}{R \cdot Re}} \beta^{\frac{1}{2}} \, d\left(\frac{Z}{R \cdot Re}\right)$$

may be integrated directly from zero to very small value of $\frac{z}{R.Re}$ and by further numerical integration using Simpson's Rule to the point where flow is almost fully developed.

With the relationship of X vs. $\frac{z}{R_o Re}$ established for values of flow behavior index "n" of 1/4, 1/2, 3/4, numerical values of Nu_z were obtained from Equation (5-41). Figs. 4-b, 4-c, 4-d, give the local Nu_z as a function of $\frac{z}{R.Re}$ for n = 1/4, 1/2, 3/4 respectively with Pr as parameter. The Prandtl number ranges from 5 to 200.

The average Nu_{avg} is found to be

$$Nu_{avg} = \frac{\int_0^{z^*} Nu_z \, dz}{\int_0^{z^*} dz^*} \tag{5-48}$$

Figs. 5-b, 5-c, 5-d, give the average Nusselt number as function of $\frac{z}{R.Re}$ for n = 1/4, 1/2, 3/4 respectively with Pr as parameter.

CHAPTER VI

DISCUSSION AND CONCLUSION

The asymptotic solution of laminar convective heat transfer, in the entrance region of a circular conduit where velocity and temperature profiles are developing simultaneously, has been obtained. Numerical values of local and average Nusselt numbers as functions of Prandtl number and dimensionless distance have been evaluated and presented in graphical forms.

It is interesting to compare the results obtained from this work with those given by reference (6). Such a comparison was made in Fig. 6 for fluids with Pr = 5. They are shown to be reasonably close to each other with the results from the present work giving a higher value of Nu_{avg} for lower values of $\frac{z}{R.Re}$ while the opposite trend being observed for higher values of $\frac{z}{R.Re}$. It may be argued that because of the omission of the convection term $V_r \frac{\partial T}{\partial r}$ the results of references (1) and (13) are less accurate for small values of $\frac{z}{R.Re}$. On the other hand, the approximation of velocity profile within the thermal boundary layer by a linear expression [Equation (3-1)] and the simplification involved in obtaining Equation (3-18) from Equation (3-15) are less justifiable when the thermal boundary layer thickness becomes significant.

The non-Newtonian fluids studied in this work were those of the pseudoplastic type obeying the "power law". By observation from Fig. 4 and 5, it may be concluded that for a fixed value of Z^*, the Nusselt number increases with increasing Prandtl number, but decreases with increasing "n". For each Prandtl number a relationship between Z^* and Nu was obtained in the form $Nu = bZ^{*a}$, with "a" and "b" constants, by approximating each curve by two straight lines. One line approximates the curve for the region $Z^* > 1.5 \times 10^{-3}$ while the other for the region $Z^* < 1.5 \times 10^{-3}$. These results may be found in Tables 9-a and 9-b.

It should be remembered that the present solution is an asymptotic one for large Prandtl numbers. For actual application of the present result, the values of Pr of the fluid has to be reasonably high in order to satisfy the assumptions made in this work and ensure the convergence of Equation (3-40). It is speculated that the solution given in this work cannot be applied to cases of $Pr < 5$ without incurring large error. The approximate method of solution employed here offers the distinct advantage of being much less difficult than the time consuming and laborious exact method. No effort is made to compare the theoretical results with those obtained by experimental work, since no such findings have ever been published.

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NOMENCLATURE

Equation numbers given after description refer to equations in which the symbols are first used or thoroughly defined. Dimensions are given in terms of mass (M), length (L), time (t), and temperature (T).

$A_o, A_1 \ldots A_k$ = coefficients defined by Equation (3-37).

$B_o, B_1 \ldots B_k$ = coefficients defined by Equation (3-36).

$C_1, C_2, C_3 \& C_4$ = coefficients of the polynomial expression for velocity distribution, Equation (5-12).

C_p = heat capacity at constant pressure, L^2/t^2T.

g = heat flux.

h = heat transfer coefficient, or increment.

$I_n(x) = i^{-n}J_n(ix)$ when J_n is the n^{th} order Bessel function.

K_1 = constant, Equation (5-23).

K_2 = constant, Equation (5-24).

K_3 = constant, Equation (5-25).

K_4 = constant, Equation (5-26).

k = thermal conductivity, ML/t^3T.

M = consistency index, Equation (1-3).

Nu_z = the local Nusselt number, Equation (3-40).

Nu_{avg} = the average Nusselt number, Equation (3-41).

n = flow behavior index, Equation (1-3).

p = fluid pressure, M/Lt^2.

Pr = the Prandtl number defined as C_p /k for Newtonian fluid.

= the Prandtl number defined as $(C_p \rho D\overline{U})/kRe_d$ for non-Newtonian fluid.

Re = the Reynolds number defined as $\overline{U}\rho R/\mu$

Re_d = the Reynolds number defined as $\rho D^n \overline{U}^{2-n}/M$.

R = radius of pipe, L.

r = radial distance, L.

r^+ = dimensionless radial distance defined by Equation (3-5).

T = temperature of fluid, T.

T_o = initial temperature of fluid, T.

T_w = temperature of wall, T.

T^+ = dimensionless temperature defined by Equation (3-8).

$T_o^+, T_1^+ \ldots . T_k^+ ..$ = functions related to T^+ by Equation (3-18).

u = dimensionless variable defined by Equation (3-17).

U = velocity of fluid in the core, L/t.

$\overline{U}$ = average velocity of fluid, L/t.

U^* = dimensionless velocity component defined by Equation (5-29).

U = velocity of fluid at entrance point, L/t.

V_r = radial component of velocity, L/t.

V_z = z-component of velocity, L/t.

V^+ = dimensionless velocity defined by Equation (3-7).

y^+ = dimensionless distance defined by Equation (3-5).

X = dimensionless parameter defined by Equation (3-35).

z = longitudinal distance, L.

z^+ = dimensionless longitudinal distance defined by Equation (3-6).

Z^* = dimensionless distance defined by $z/R.Re_d$.

α = $k/\rho C_p$, thermal diffusivity, L^2/t.

β = dimensionless parameter defined by Equation (3-9).

γ = a parameter appearing in Equation (4-3).

δ = velocity boundary layer thickness, L.

δ^* = δ/R, dimensionless velocity boundary layer thickness.

$\Gamma(x)$ = gamma function.

ρ = density, M/L^3.

η = dimensionless variable defined by Equation (3-25).

Θ = parameter defined by Equation (3-16).

ζ = dimensionless variable defined by Equation (3-29).

μ = viscosity of fluid, M/Lt.

τ_o = parameter in Bingham model, Equation (1-1).

γ = kinematic viscosity of fluid.

μ_o = parameter in Bingham model, Equation (1-1).

τ_{rz} = shear stress exerted in the z-direction and perpendicular to the r-direction.

APPENDIX A

SAMPLE CALCULATION FOR DETERMINATION OF COEFFICIENTS C_i

IN EQUATION (5-12)

1. By matching Equation (5-12) with Equation (5-14)

Equation (5-12) is given as

$$\frac{V_z}{U} = C_1\left(\frac{y}{\delta}\right) + C_2\left(\frac{y}{\delta}\right)^2 + C_3\left(\frac{y}{\delta}\right)^3 + C_4\left(\frac{y}{\delta}\right)^4 \qquad \text{(A-1)}$$

If n = 1/2, Equation (5-14) is equal to

$$\frac{V_z}{U} = 1-\left(1-\frac{y}{\delta}\right)^3$$

$$= 3\left(\frac{y}{\delta}\right) - 3\left(\frac{y}{\delta}\right)^2 + \left(\frac{y}{\delta}\right)^3 \qquad \text{(A-2)}$$

Comparing with Equation (5-12), we have

$$C_1 = 3$$
$$C_2 = -3$$
$$C_3 = 1$$
$$C_4 = 0$$

2. By Equations (5-17) to (5-20), Equations (5-17), (5-18) and (5-19) can be rearranged to the form

$$C_1 + C_2 + C_3 = 1 - C_4 \tag{A-3}$$

$$C_1 + 2C_2 + 3C_3 = -4C_4 \tag{A-4}$$

$$\frac{C_1}{3} + \frac{C_2}{6} + \frac{C_3}{10} = 1 - \frac{2n}{1 + 3n} - \frac{C_4}{15} \tag{A-5}$$

One can obtain expressions for C_1, C_2 and C_3 in terms of C_4 and "n" which are written as follows:

$$C_1 = 7 - \frac{20n}{1 + 3n} - \frac{1}{3} C_4 \tag{A-6}$$

$$C_2 = -11 + \frac{40n}{1 + 3n} + \frac{5}{3} C_4 \tag{A-7}$$

$$C_3 = 5 - \frac{20n}{1 + 3n} - \frac{7}{3} C_4 \tag{A-8}$$

Substitution of Equations (A - 6), (A-7) and (A-8) into (5-19) gives C_4 in terms of "n":

$$C_4 = 3780 \left\{ - \frac{1}{252} \pm \sqrt{\left(\frac{1}{252}\right)^2 + \frac{1}{1890}\left[- \frac{65}{840} + \frac{5n}{21(1 + 3n)} - \frac{50n^2}{21(1 + 3n)^2} + \frac{n}{4n + 2}\right]} \right\} \tag{A-9}$$

Therefore, once the value of "n" is chosen, C_4 is first obtained by Equation (A-9), then C_1, C_2 and C_3 can be found by Equations (A-6), (A-7) and (A-8).

For example, if $n = 1/4$, we have

$$C_4 = 3780 \left\{ \left(-\frac{1}{252}\right) \pm \sqrt{\left(\frac{1}{252}\right)^2 + \frac{1}{1890} - \frac{65}{840} + \frac{5(.25)}{21(1+.75)} - \frac{50(.25)^2}{21(1+.75)^2} + \frac{.25}{3}} \right\}$$

$$= -2.35896 \qquad \text{(A-10)}$$

$$C_3 = 5 - \frac{20(.25)}{1+.75} - \frac{7}{3}(-2.35896) = 7.64709 \qquad \text{(A-11)}$$

$$C_2 = -11 + \frac{40(.25)}{1+.75} + \frac{5}{3}(-2.35896) = -9.21731 \qquad \text{(A-12)}$$

$$C_1 = 7 - \frac{20(.25)}{1+.75} - \frac{1}{3}(-2.35896) = 4.92917 \qquad \text{(A-13)}$$

APPENDIX B

NUMERICAL INTEGRATION OF EQUATION (5-38)

Sample calculation is based on $n = 1/4$, and Equation (5-38) becomes

$$\frac{dZ^*}{dU^*} = \left\{ \frac{0.340421}{U^*} \left(1 - \sqrt{1 - 1.775061\left(1 - \frac{1}{U^*}\right)}\right) \right\}^{1/4} \left\{ 0.928311\, U^* \right.$$

$$- 0.849799 + 0.271287\, U^* \sqrt{1 - 1.775061\left(1 - \frac{1}{U^*}\right)}$$

$$\left. - \frac{0.120388}{\sqrt{1 - 1.775061\left(1 - \frac{1}{U^*}\right)}} \right\} \qquad \text{(B-1)}$$

The solution is required over the interval $U^* = 1$ to U^* = fully developed value which is 1.4000 by Equation (5-40) given that $U^* = 1$ at $Z^* = 0$. Simpson's rule may be written as:

$$Z^*(U_2^*) = Z^*(U_o^*) + \frac{h}{3}\, Z^{*\prime}(U_o^*) + 4Z^{*\prime}(U_1^*)$$

$$+ Z^{*\prime}(U_2^*) \qquad \text{(B-2)}$$

From the method of Clippenger and Dimsdale (8)

$$Z^{*}(U_1^{*}) = \frac{1}{2}\, Z^{*}(U_o^{*}) + Z^{*}(U_2^{*}) + \frac{h}{4}\, Z^{*\prime}(U_o^{*})$$

$$- Z^{*\prime}(U_2^{*}) \tag{B-3}$$

Where

$Z^{*}(U_i^{*})$ = value of Z^{*} when U^{*} is equal to U_i^{*}

$Z^{*\prime}(U_i^{*})$ = value of $\frac{dZ^{*}}{dU^{*}}$ when U^{*} is equal to U_i^{*}

h = increment of U^{*}

The first few results of Equation (B - 1) are given in Table B.1 where h is chosen to be 0.01.

Table B.1 Partial Solution of Equation (B-1)

U^{*}	$\frac{dZ^{*}}{dU^{*}}$	Z^{*}
1.00	0	0
1.01	.0555643	3.13620×10^{-4}
1.02	.0681703	9.68092×10^{-4}
1.03	.0777347	1.69872×10^{-3}

The computing procedures are:

1. From Equation (B-1), find $\frac{dZ^{*}}{dU^{*}}$ at U^{*} = 1.0, 1.01, 1.02, etc.

The value of $\frac{dZ^*}{dU^*}$ at $U^* = 1.00$ is 0

at $U^* = 1.01$ is 0.055564

at $U^* = 1.02$ is 0.068170

2. Use Equation (B-2) to find $Z^*(1.02)$, which is 9.6809×10^{-4}. Values for $Z^*(1.04)$, $Z^*(1.06)$, $Z^*(1.08)$......................, are found in the same manner.

3. To find $Z^*(1.01)$, Equation (B-3) is applied in conjunction with the results obtained from Equation (B-1) and Equation (B-2).

From Equation (B-1) the value $Z^{*'}(U_o^*) = 0$

$$Z^{*'}(U_2^*) = 0.068170$$

From Equation (B-2), $Z^*(U_o^*) = 0$ and $Z^*(U_2^*) = 9.6809 \times 10^{-4}$. Thus $Z^*(U_1^*)$ can be obtained and is, by Equation (B-3), found to be 3.13620×10^{-4}. Values for $Z^*(1.03)$, $Z^*(1.05)$, $Z^*(1.07)$..........., are found in like manner.

Actual computation was carried out by an LGP-30 digital computer. The results are tabulated in Table 2 and presented graphically in Fig. 9 and Fig. 10.

APPENDIX C

SOLUTION OF EQUATION (5-38) FOR VERY SMALL VALUES OF Z^*

Equation (5-38) may be written as follows:

$$K_1^n\left[1 - \sqrt{1 + \frac{2K_2}{K_1^2}(1 - \frac{1}{U^*})}\right]^n \Bigg\{ U^*(\frac{1}{2} - \frac{K_1K_3}{2} - \frac{K_4K_1^2}{K_2^2} - \frac{K_4}{K_2})$$

$$+ (\frac{K_4}{2K_2} - \frac{1}{2}) + \frac{K_1}{K_2} \quad U^*\sqrt{1 + \frac{2K_2}{K_1^2}(1 - \frac{1}{U^*})} \; (K_3 + \frac{K_4K_1}{K_2})$$

$$+ \frac{1}{2\sqrt{1 + \frac{2K_2}{K_1^2}(1 - \frac{1}{U^*})}} \left[\frac{K_3}{K_1} + \frac{K_4}{K_2}\right]\Bigg\} \frac{dU^*}{U^{*n}}$$

$$= \frac{d(\frac{z}{R\rho D^n U^{2-n}})}{M} \qquad \text{(C-1)}$$

For very small Z^* then $U^* \approx 1$.

Thus Equation (C-1) may be reduced to

$$\frac{K_1^n}{C_1^n(-K_2)^n Z^{2n}}\left[(\frac{1}{2} - \frac{K_1K_3}{K_2} - \frac{K_4K_1^2}{K_2^2} - \frac{K_4}{K_2}) + \frac{K_1}{K_2}(K_3 + \frac{K_4K_1}{K_2})\right.$$

$$\left. + \frac{1}{2}(\frac{K_3}{K_1} + \frac{K_4}{K_2})\right]\left[1 - \sqrt{1 + \frac{2K_2}{K_1^2}(1 - \frac{1}{U^*})}\right] dU^*$$

$$= dZ^* \qquad \text{(C-2)}$$

Now approximating the term

$$1 - \sqrt{1 + \frac{2K_2}{K_1^2}(1 - \frac{1}{U^*})}$$

in Equation (C-2) by

$$- \frac{K_2}{K_1^2}(U^* - 1)$$

then Equation (C-2) becomes

$$P(U^* - 1)^n \, dU^* = dZ^* \tag{C-3}$$

where

$$P = (- \frac{K_2}{K_1^2})^n \left\{ (- \frac{K_1}{4C_1K_2}) \left[(\frac{1}{2} - \frac{K_1K_3}{K_2} - \frac{K_4K_1^2}{K_2^2} - \frac{K_4}{K_2}) \right.\right.$$

$$\left.\left. + (\frac{K_4}{2K_2} - \frac{1}{2}) + \frac{K_1}{K_2}(K_3 + \frac{K_4K_1}{K_2}) + \frac{1}{2}(\frac{K_3}{K_1} + \frac{K_4}{K_2}) \right] \right\} \tag{C-4}$$

Integration of (C-3) becomes

$$Z^* = \frac{P}{n + 1}(U^* - 1)^{n+1} \tag{C-5}$$

or

$$(U^* - 1)^{n+1} = (\frac{n + 1}{P})Z^* \tag{C-6}$$

APPENDIX D

SAMPLE CALCULATION OF LOCAL NUSSELT NUMBER FOR NON-NEWTONIAN FLUID

In order to find Nu_z at a certain value of Z_2^*, the integral $\int_0^{Z_2^*} \beta^{1/2} dZ^*$ must be determined first, where Z_1^* is a very small value.

From Equation (5-47) one can obtain

$$\beta^{1/2} = \sqrt{2C_1\left(\frac{p}{n+1}\right)^{n+1}} \quad Z_1^{*\,-1/2(n+1)} \tag{D-1}$$

Thus for a very small value of Z^* the integral $\int_0^{Z_1^*} \beta^{1/2} dz^*$ may be determined by direct integration to Z_1^*. To evaluate the integral at Z_2^* where $Z_2^* > Z_1^*$ then graphical integration is employed using values of β obtained from the developing velocity profile data presented in Table 2.

For example if $n = 1/2$ then by Equation (D-1) $\beta^{1/2} = 1.0662\, Z^{*\,-1/3}$ and integration of $\int^{z^*} \beta^{1/2} dz^*$ from $Z^* = 0$ to $Z_1^* = 10^{-6}$ gives value of 2.927×10^{-4}. Now if Nu_z is to be determined at Z_2^* then the integral

$$\int_0^{Z_2^*} \beta^{1/2} dz^* = \int_0^{Z_1^*} \beta^{1/2} dz^* + \int_{10^{-6}}^{z^*} \beta^{1/2} dz_2^* = 2.927 \times 10^{-4} + \int_{10^{-6}}^{Z_2^*} \beta^{1/2} dz^*$$

must be evaluated.

The integral $\int_{10^{-6}}^{Z_2^*} \beta^{1/2} dZ^*$ is found by graphical integration. The figure below shows a schematic diagram of $\beta^{1/2}$ vs. Z^*.

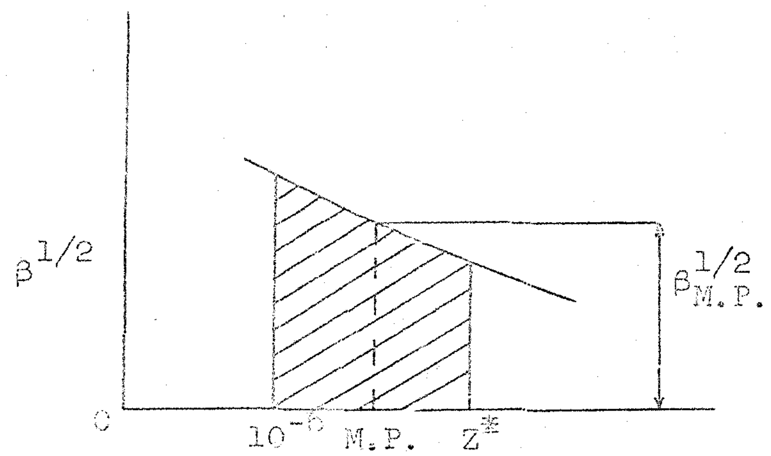

Fig. D-1 Schematic Diagram of $\beta^{1/2}$ vs. Z^*

where M.P. is the mid-point between 10^{-6} and Z^*.

Therefore,

$$\int_0^{Z^*} \beta^{1/2} dZ^* \approx \text{area of shaded section} \approx \beta^{1/2}(Z^* - 10^{-6}) \qquad \text{(D-3)}$$

The width of the shaded section should be kept small enough to give a good approximation.

If $Z_2^* = 10^{-4}$, then one can by taking small increments of Z^*, integrate graphically to find

$$\int_{10^{-6}}^{10^{-4}} \beta^{1/2} dZ^* = 1.7775 \times 10^{-3}$$

Thus

$$\int_0^{10^{-4}} \beta^{1/2} dZ^* = 2.9270 \times 10^{-4} + 1.7775 \times 10^{-3}$$

$$= 20.702 \times 10^{-4} \text{ where } Z^* = \frac{z}{R.Re_d}$$

$$= 29.277 \times 10^{-4} \text{ where } Z^* = \frac{z}{R.Re_R}$$

Now the Nu_z may be evaluated at $Z^* = 10^{-4}$. If Pr = 5 is chosen then the parameter X given by Equation (4-1) may be calculated to be

$$X = (\frac{5}{9})^{1/3} \frac{11.2}{(29.277 \times 10^{-4})} = 64.3534$$

and $Nu_z = 71.5008$ by Equation (3-40).

APPENDIX E

FIGURES

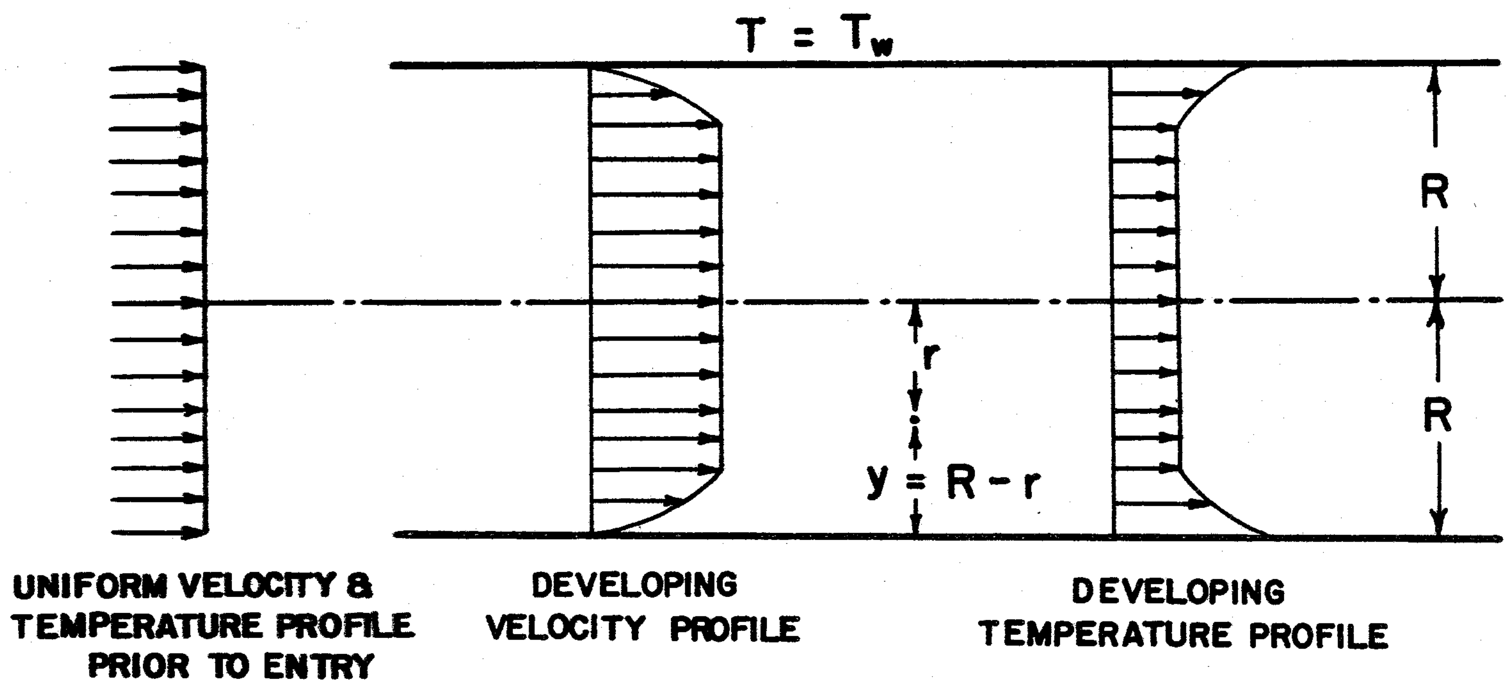

FIG. I DEVELOPMENT OF VELOCITY & TEMPERATURE PROFILES IN THE ENTRANCE REGION

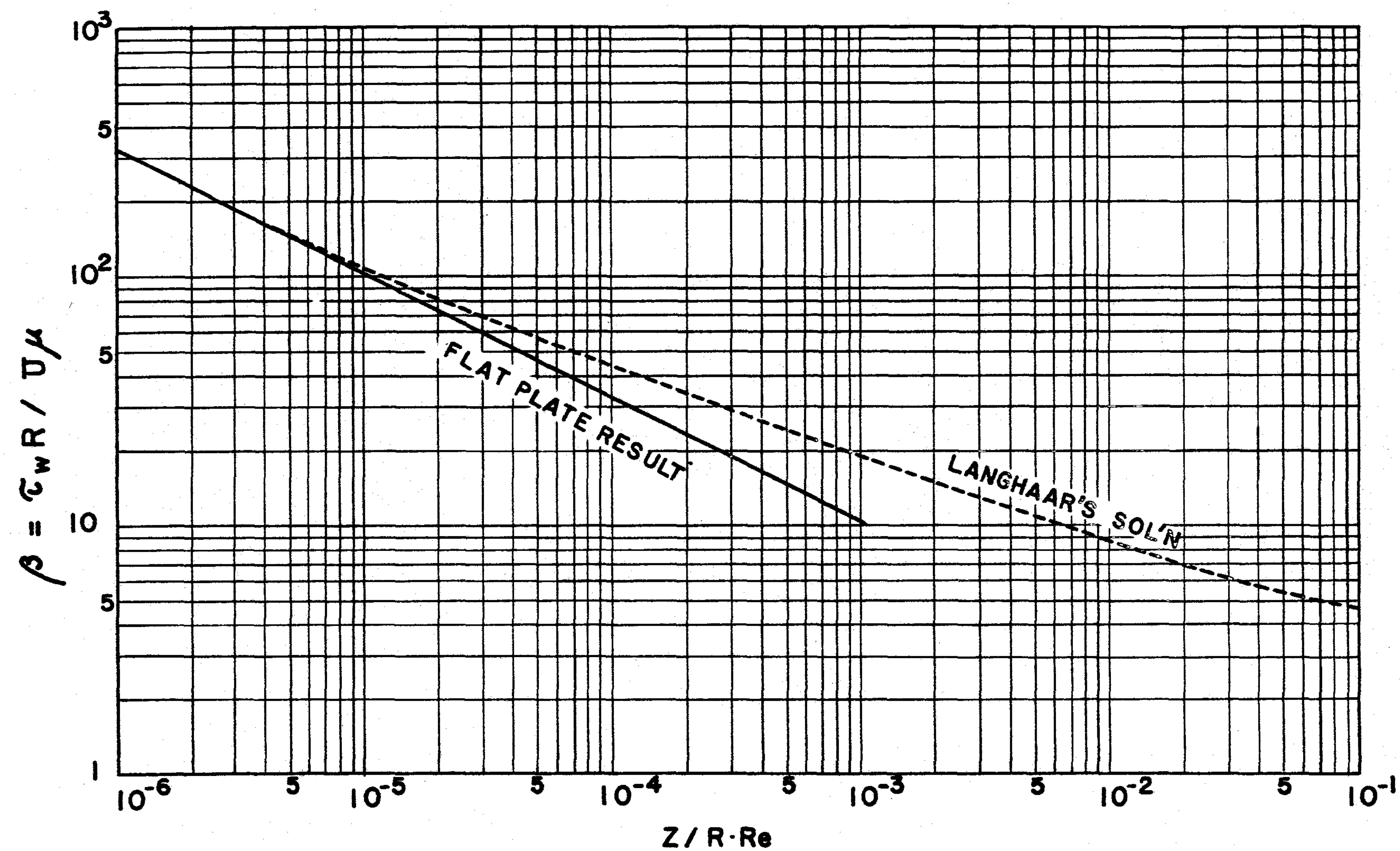

FIG. 2-a RELATIONSHIP BETWEEN β AND Z / R·Re FOR NEWTONIAN FLUID

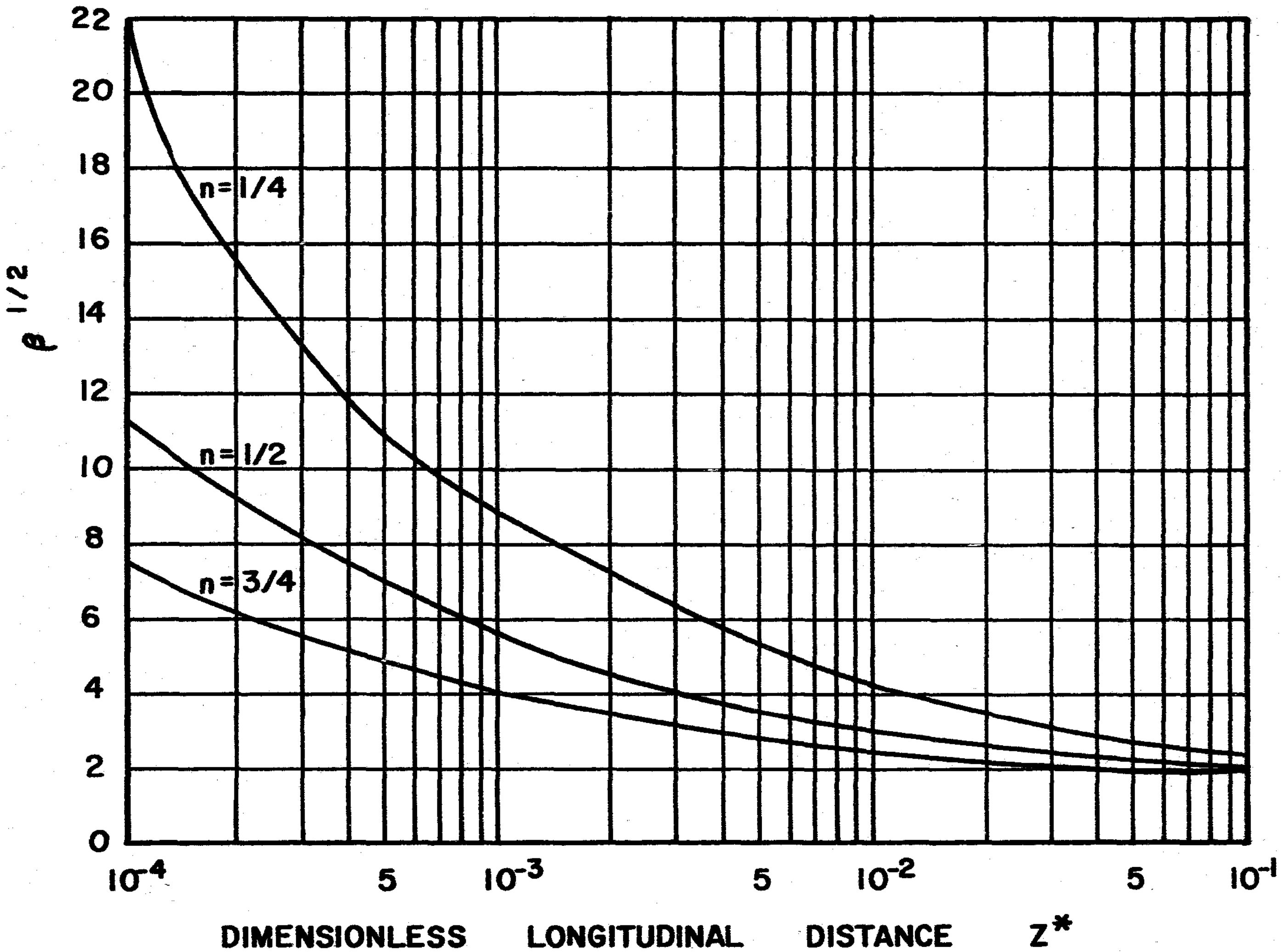

FIG. 2-b RELATIONSHIP BETWEEN $\beta^{1/2}$ AND $Z/R \cdot Re$

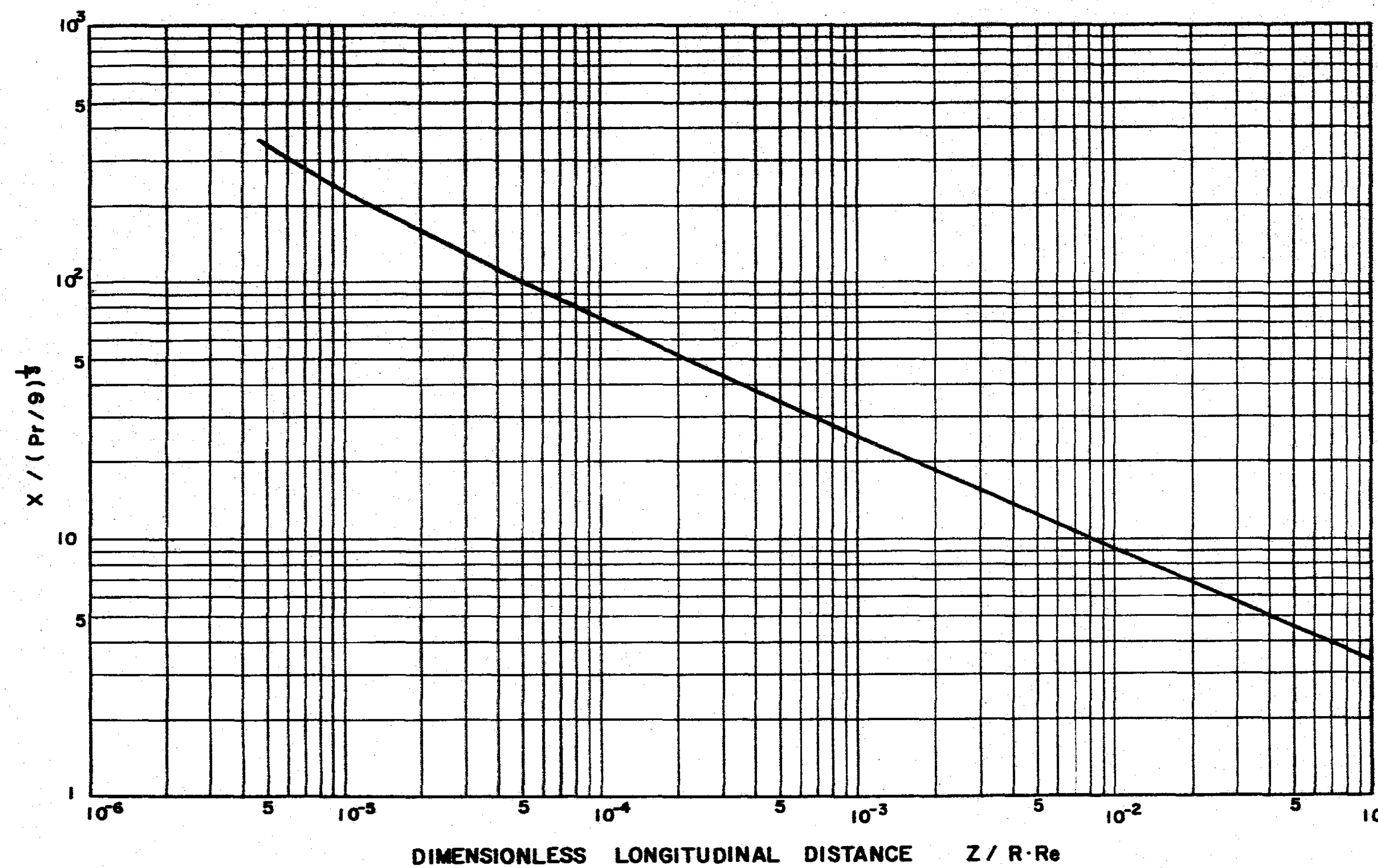

FIG. 3-a RELATIONSHIP BETWEEN $X / (Pr/9)^{\frac{1}{3}}$ AND Z/RRe FOR NEWTONIAN FLUID

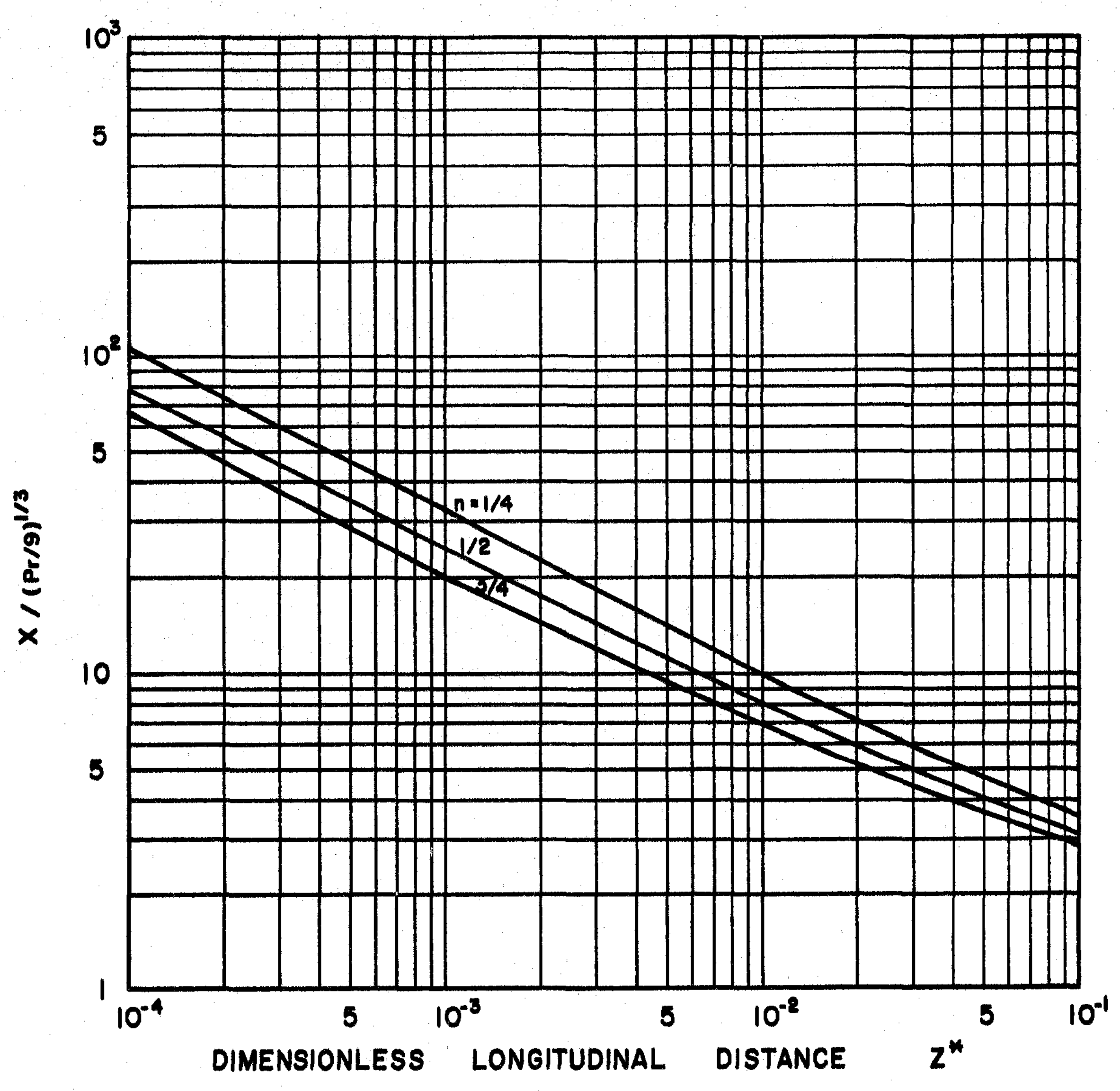

FIG. 3-b RELATIONSHIP BETWEEN $X/(Pr/9)^{1/3}$ AND $Z/R\cdot Re$

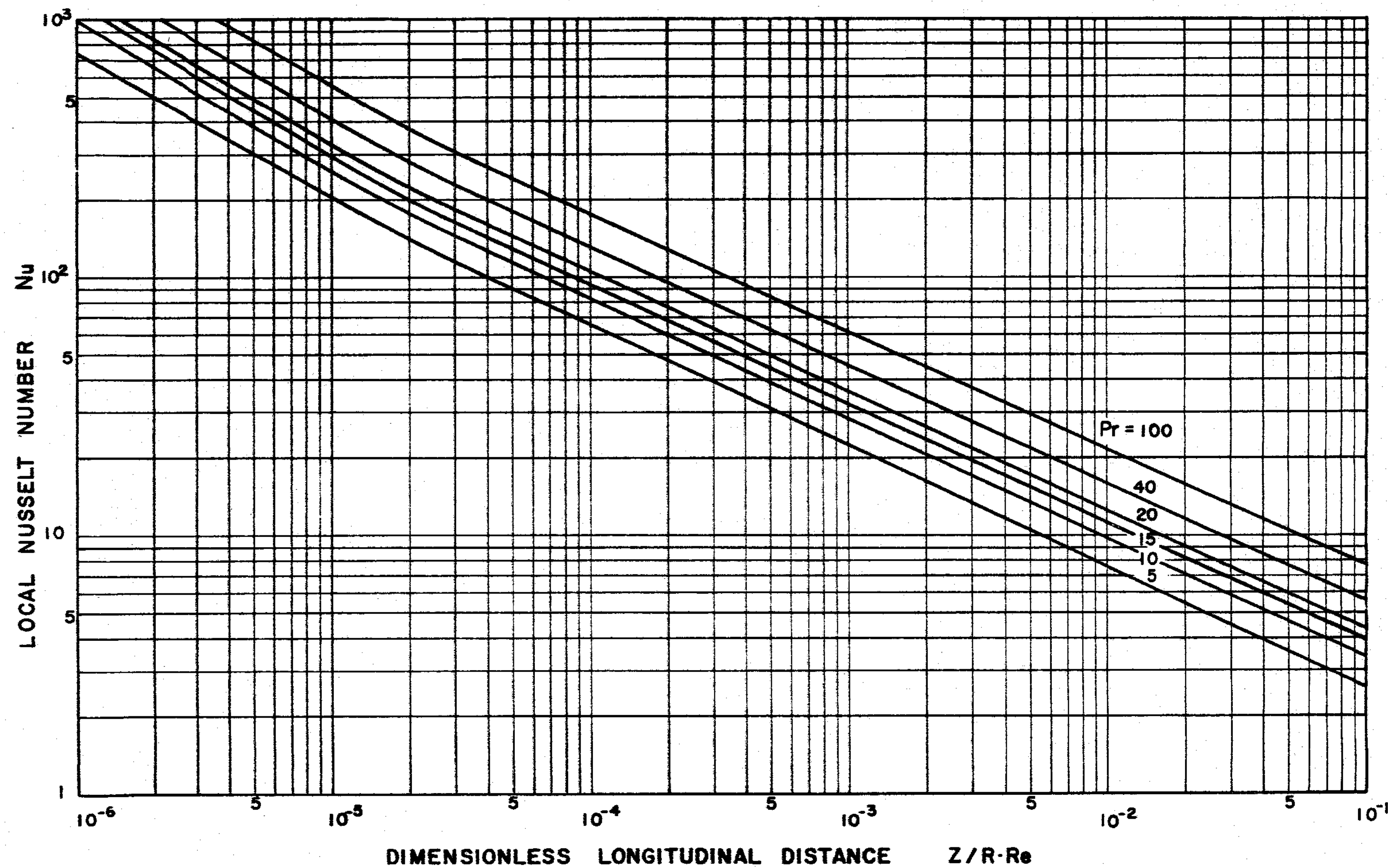

FIG. 4-a RELATIONSHIP BETWEEN LOCAL NUSSELT NUMBER Nu AND Z / R·Re FOR NEWTONIAN FLUID

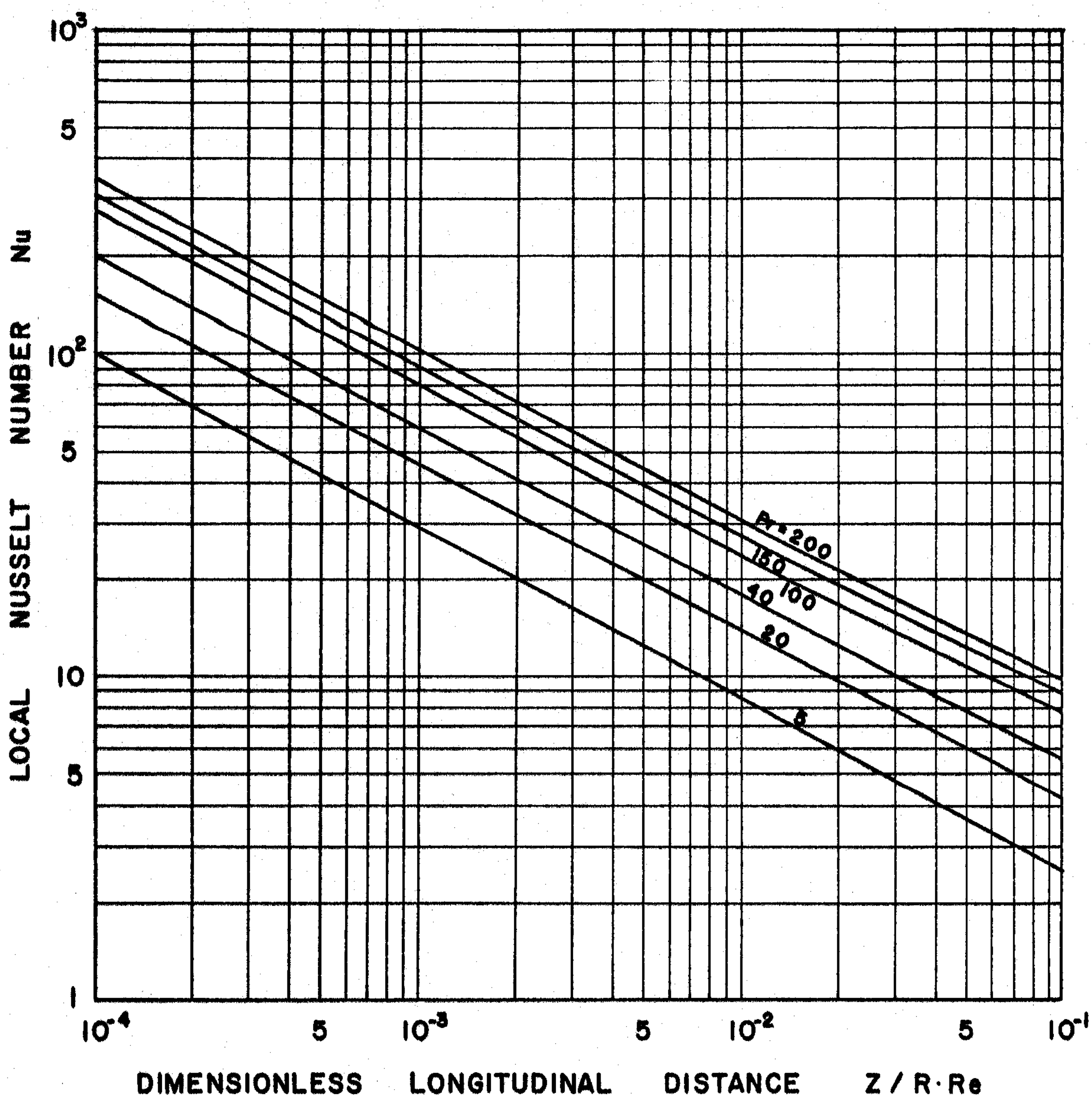

FIG. 4-b RELATIONSHIP BETWEEN LOCAL NUSSELT NUMBER Nu AND Z/R·Re FOR n = 1/4

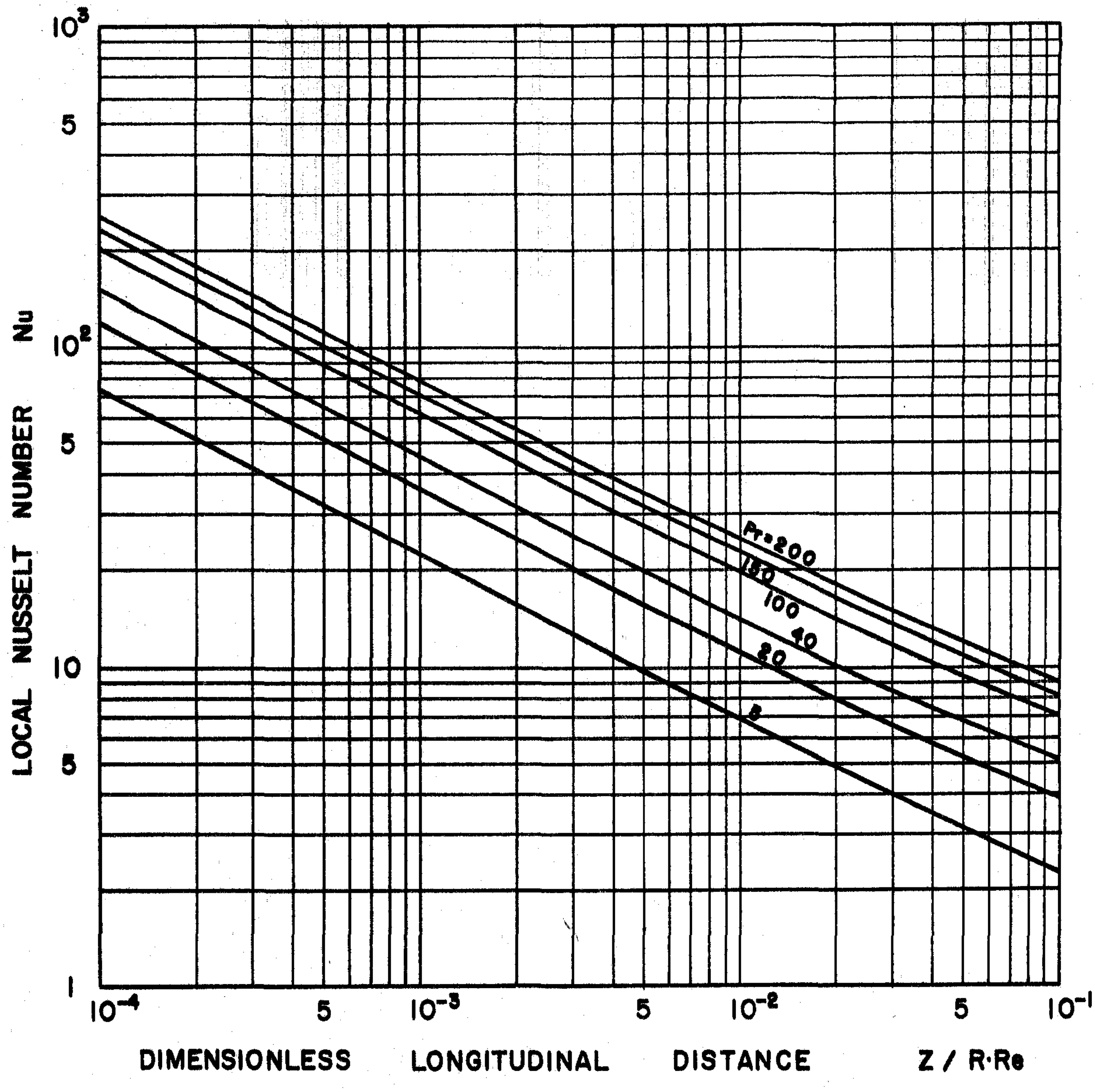

FIG. 4-c RELATIONSHIP BETWEEN LOCAL NUSSELT NUMBER Nu AND Z / R·Re FOR n = 1/2

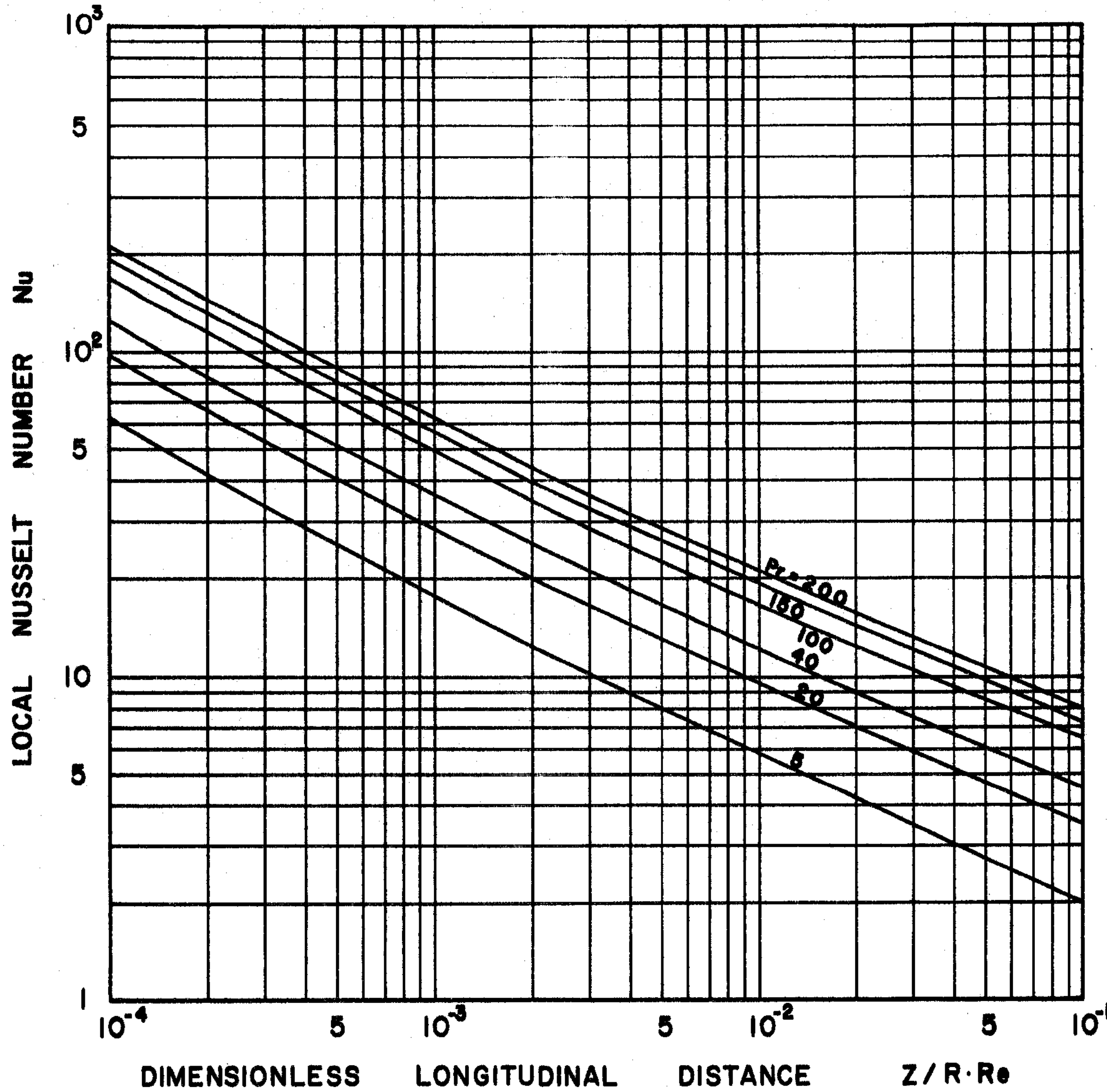

FIG. 4-d RELATIONSHIP BETWEEN LOCAL NUSSELT NUMBER Nu AND Z / R·Re FOR n = 3/4

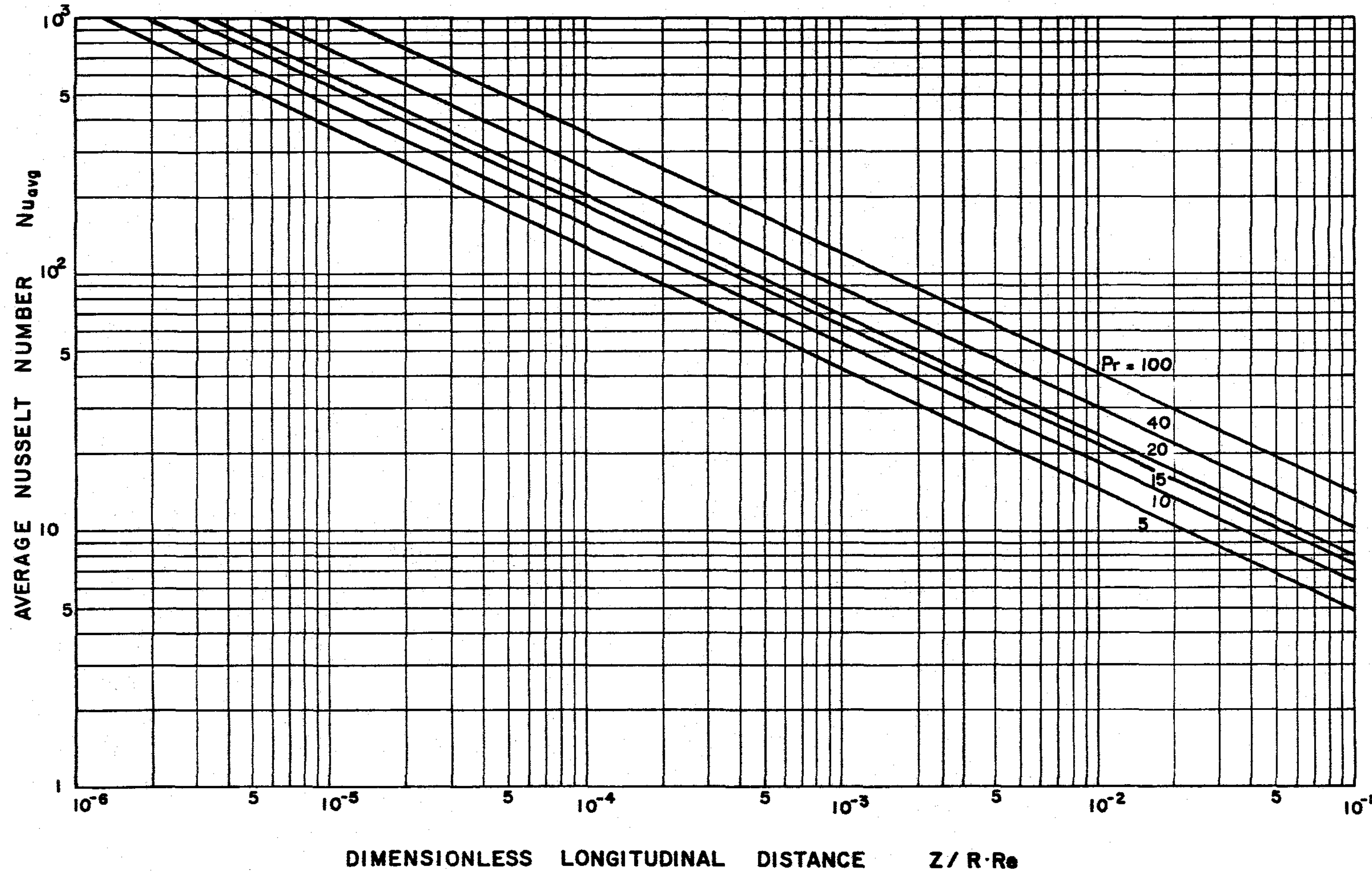

FIG. 5-a RELATIONSHIP BETWEEN AVERAGE NUSSELT NUMBER Nu_{avg} AND Z / R·Re FOR NEWTONIAN FLUID

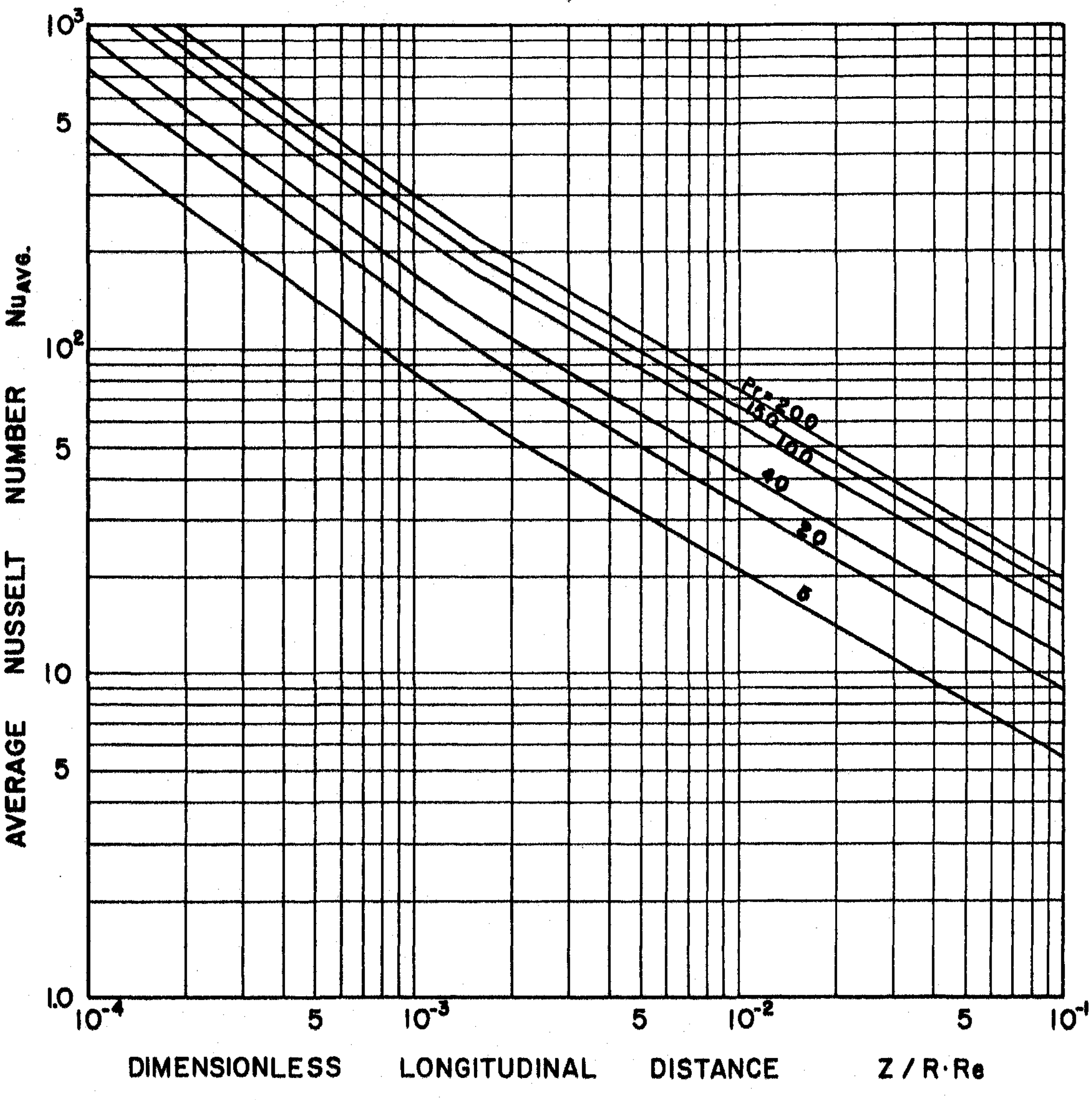

FIG. 5-b RELATIONSHIP BETWEEN AVERAGE NUSSELT NUMBER Nu_{AVG} AND $Z/R{\cdot}Re$ FOR $n = 1/4$

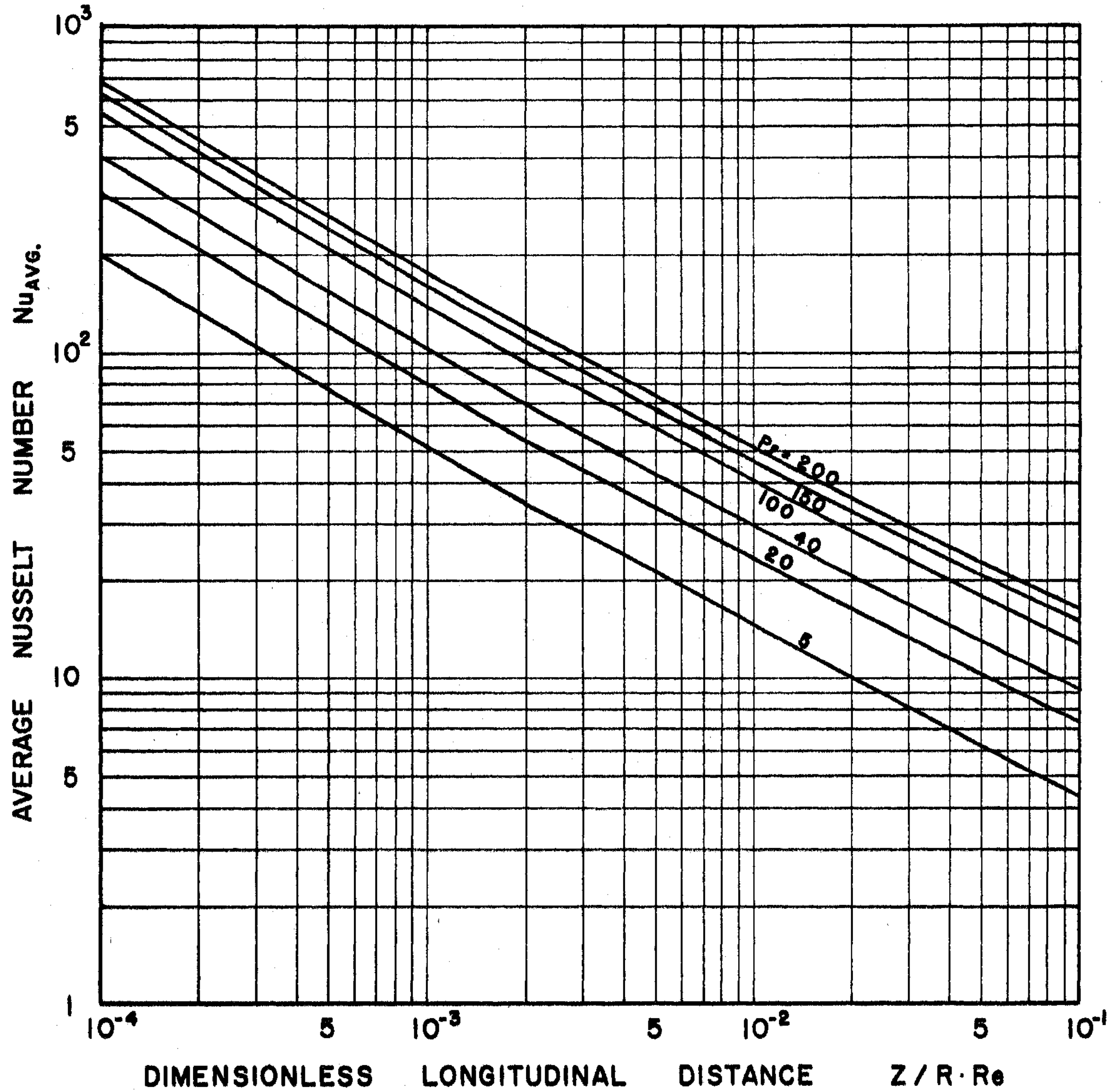

FIG. 5-c RELATIONSHIP BETWEEN AVERAGE NUSSELT NUMBER $Nu_{AVG.}$ AND $Z/R \cdot Re$ FOR $n = 1/2$

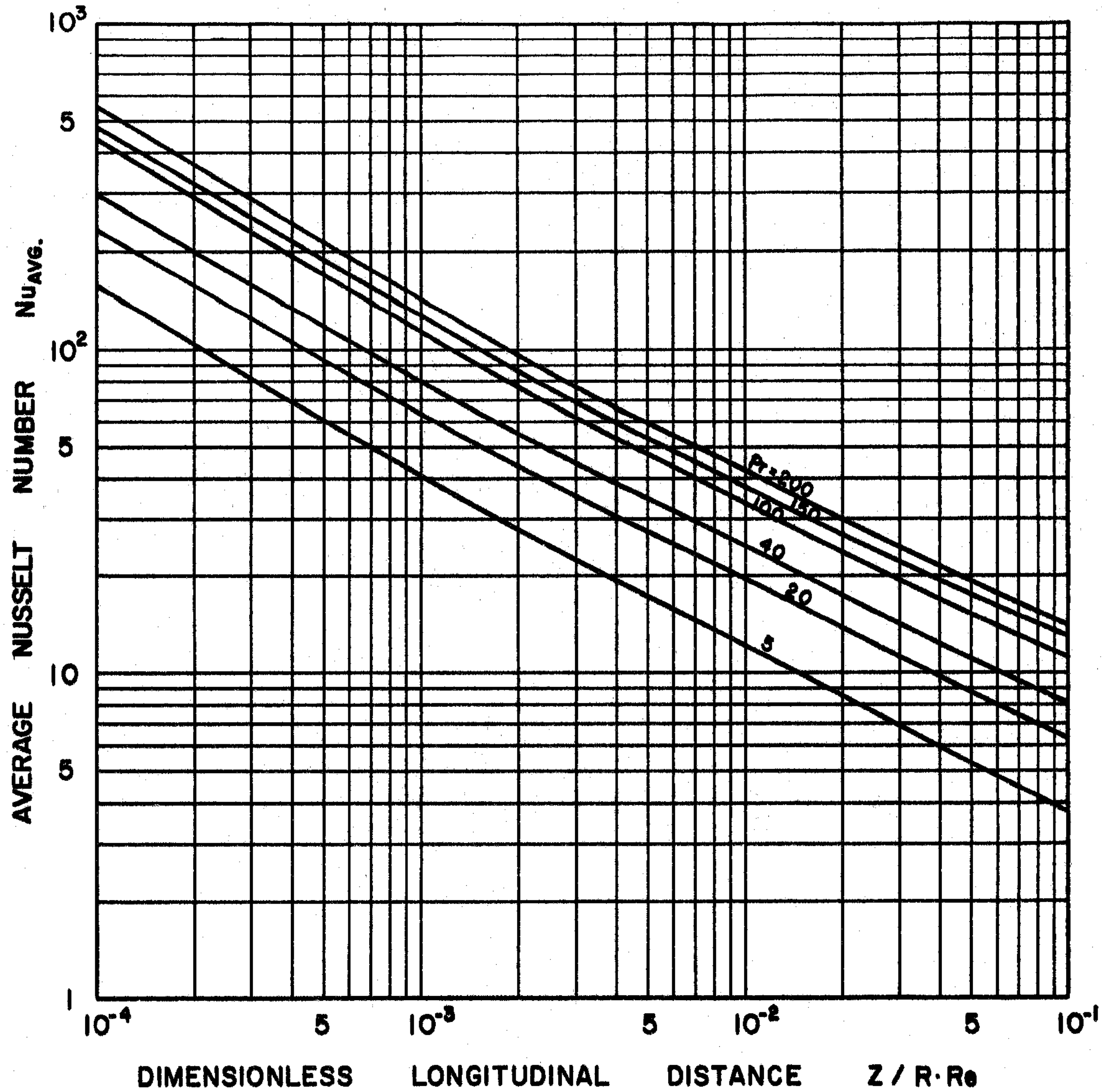

FIG. 5-d RELATIONSHIP BETWEEN AVERAGE NUSSELT NUMBER $Nu_{AVG.}$ AND $Z/R \cdot Re$ FOR $n = 3/4$

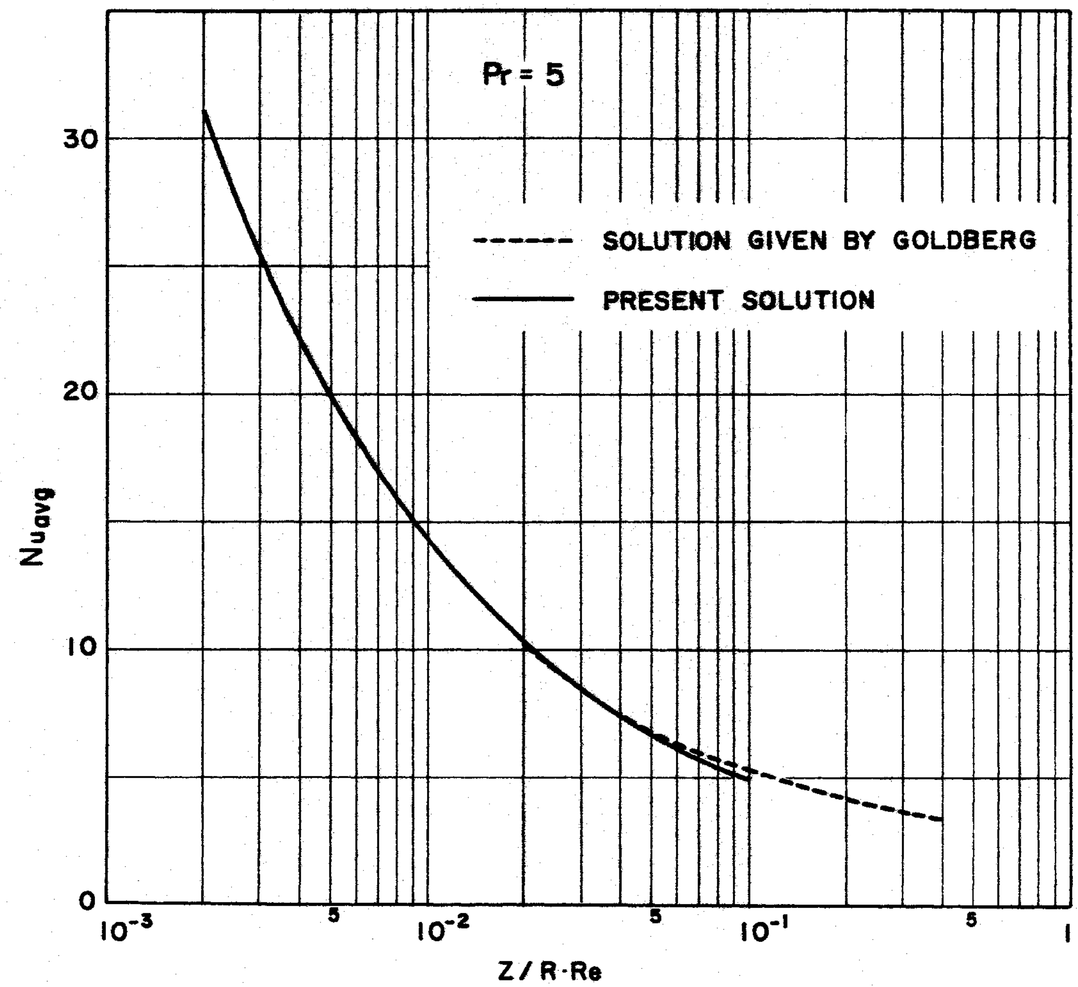

FIG. 6 COMPARISON BETWEEN THE ASYMPTOTIC SOLUTION AND GOLDBERG'S SOLUTION

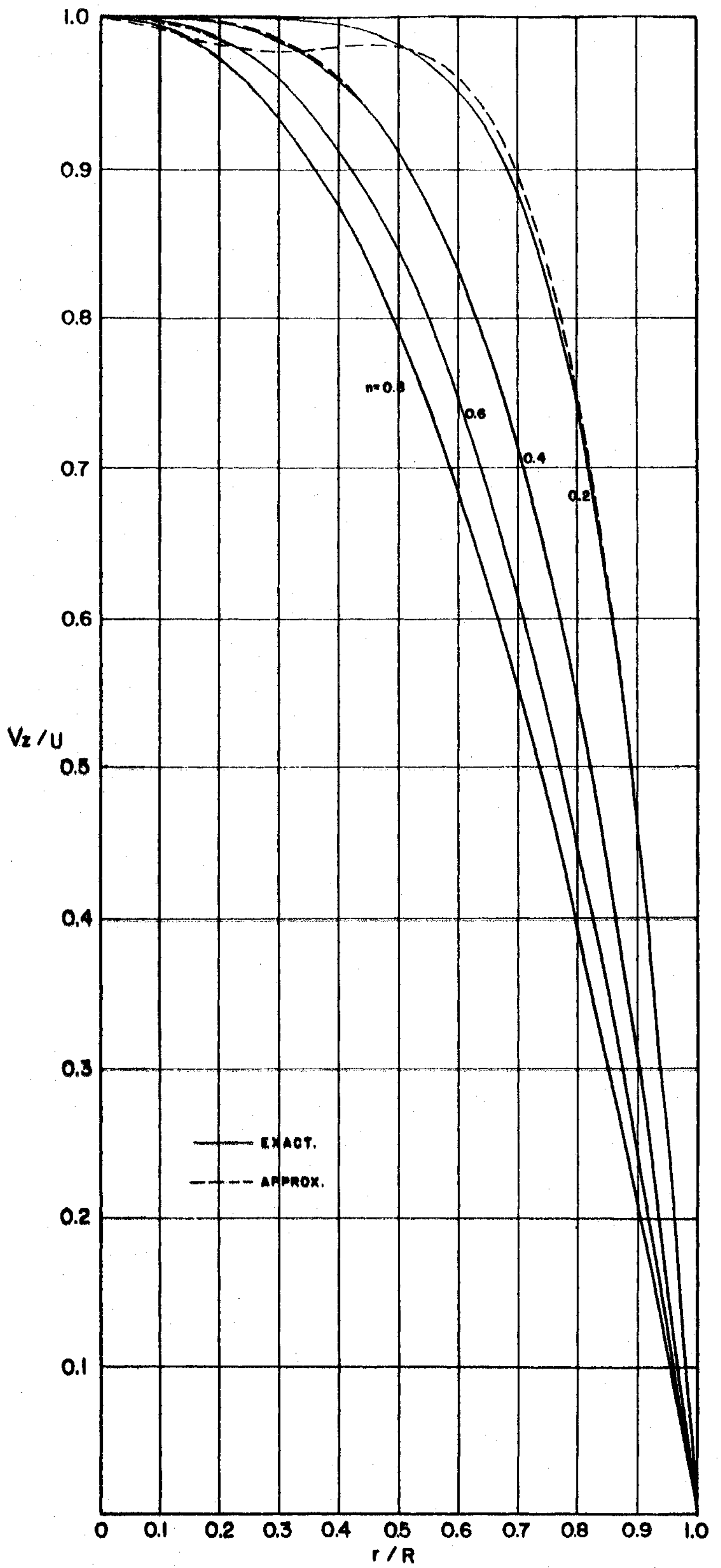

FIG. 7 COMPARISON OF EXACT AND APPROXIMATE VELOCITY PROFILES

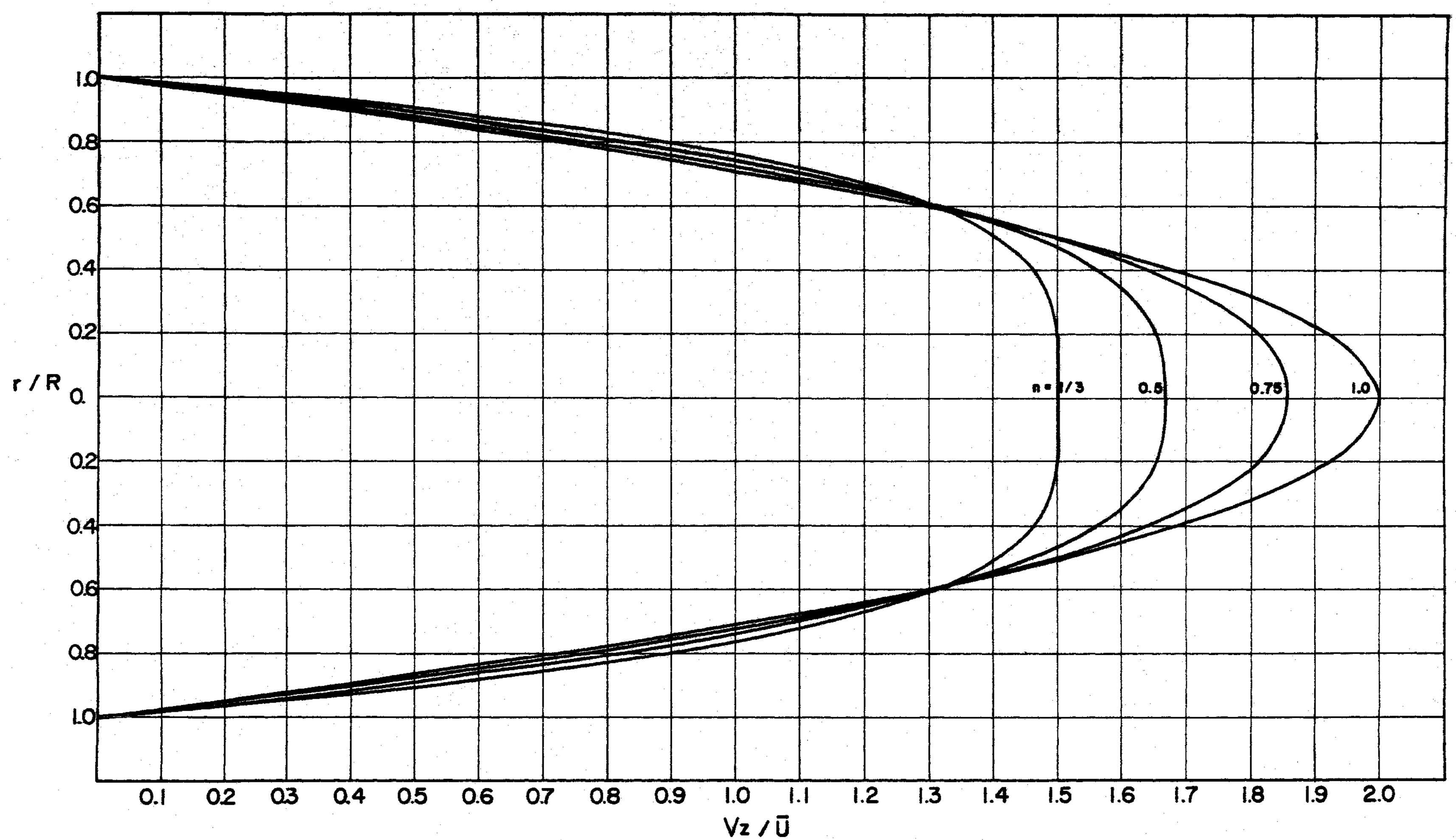

FIG. 8 FULLY DEVELOPED VELOCITY PROFILES

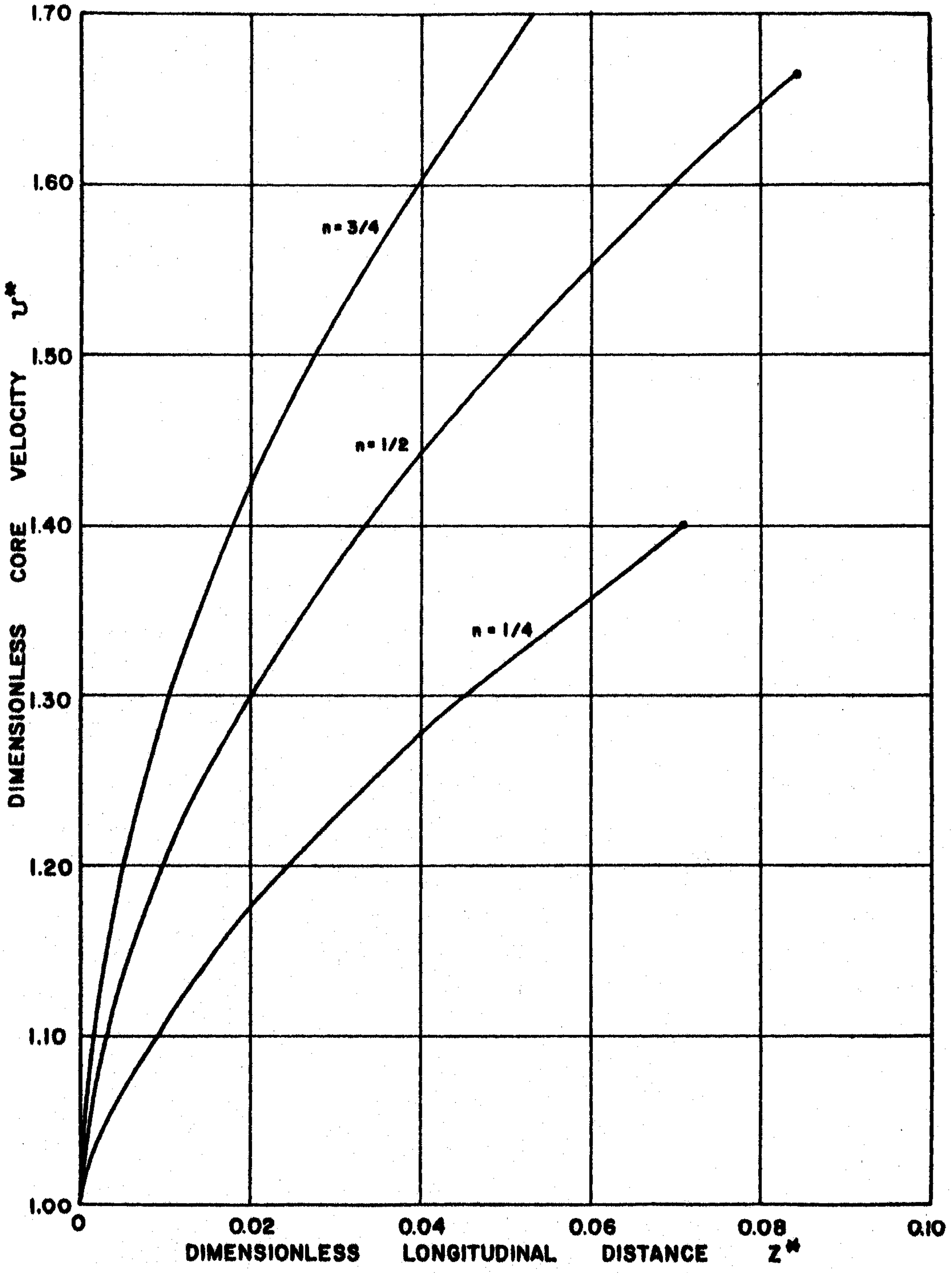

FIG. 9 DIMENSIONLESS CORE VELOCITY U* VS. DIMENSIONLESS LONGITUDINAL DISTANCE Z*

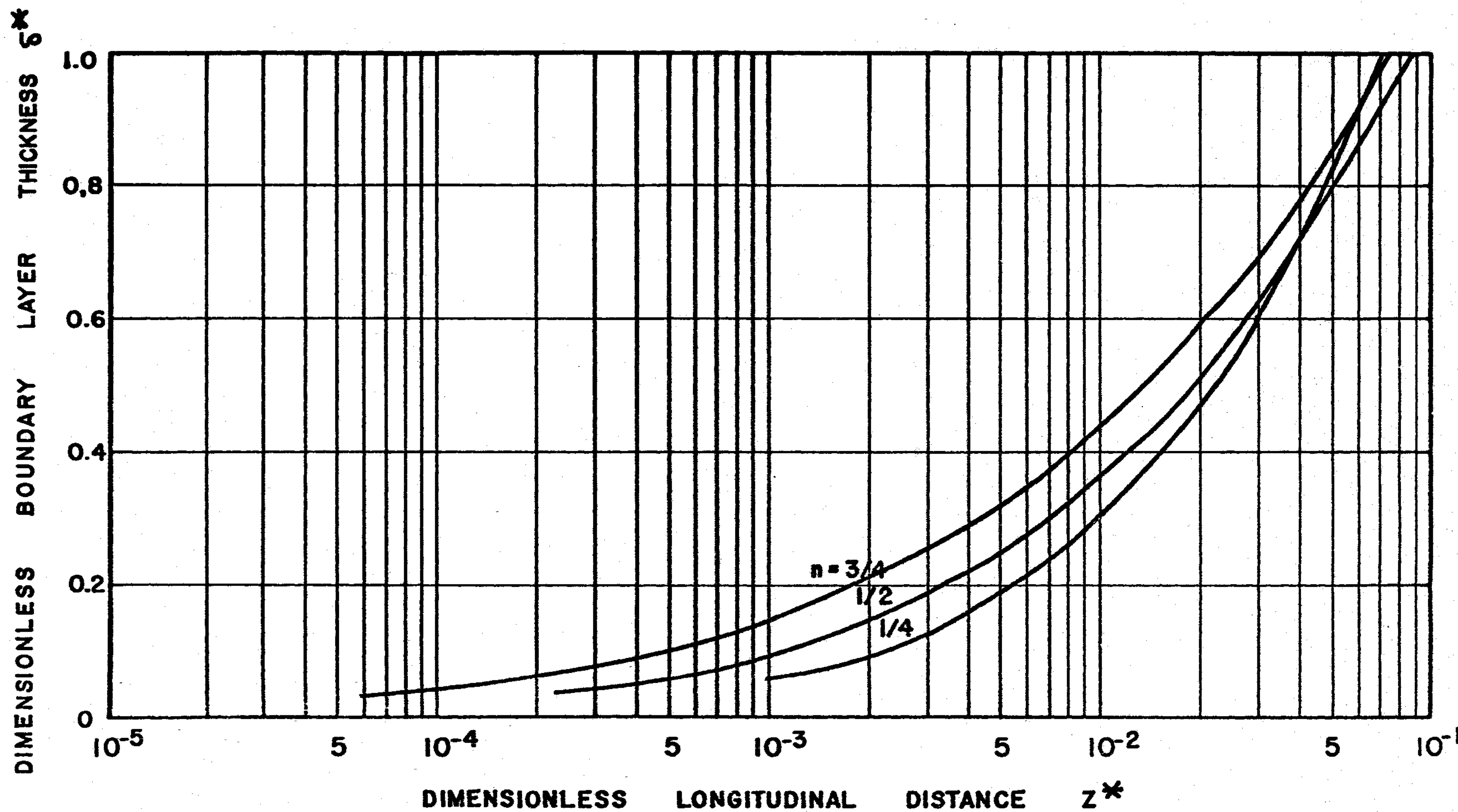

FIG. 10 DIMENSIONLESS VELOCITY BOUNDARY THICKNESS δ^* VS. DIMENSIONLESS LONGITUDINAL DISTANCE Z^*

APPENDIX F

TABLES

Table 1

The Coefficients C_i for Velocity Profile

n	C_1	C_2	C_3	C_4
1.00000	2.00000	-1.00000	0	0
0.80000	2.23813	-1.30832	-0.09776	0.16795
0.75000	2.32052	-1.44875	-0.06406	0.19229
0.60000	2.65803	-2.14727	0.32047	0.16778
0.50000	3.00000	-3.00000	1.00000	0
0.40000	3.50758	-4.44699	2.37124	-0.43183
0.33333	4.00000	-6.00000	4.00000	-1.00000
0.20000	5.79192	-12.45958	11.54342	-3.87575
0.10655	10.38180	-32.77036	38.38518	-15.00000

Table 2

Dimensionless Velocity (U^*) and Dimensionless Velocity Boundary Layer Thickness (δ^*) as Function of Dimensionless Distance (Z^*)

For $n = \frac{3}{4}$

Z^*	U^*	δ^*
0.00000000	1.00	0.00000000
0.00005929	1.02	0.03290695
0.00020804	1.04	0.06503524
0.00043676	1.06	0.09642303
0.00074368	1.08	0.12710593
0.00112855	1.10	0.15711740
0.00159165	1.12	0.18648890
0.00213347	1.14	0.21524997
0.00275453	1.16	0.24342844
0.00345535	1.18	0.27105056
0.00423637	1.20	0.29814111
0.00509797	1.22	0.32472352
0.00604047	1.24	0.35081997
0.00706409	1.26	0.37645146
0.00816902	1.28	0.40163791
0.00935532	1.30	0.42639825
0.01062305	1.32	0.45075048

Table 2 (cont'd)

For $n = \frac{3}{4}$

Z^*	U^*	δ^*
0.01197215	1.34	0.47471172
0.01340256	1.36	0.49829833
0.01491413	1.38	0.52152580
0.01650668	1.40	0.54440910
0.01817998	1.42	0.56696241
0.01993376	1.44	0.58919936
0.02176773	1.46	0.61113297
0.02368155	1.48	0.63277584
0.02567485	1.50	0.65413994
0.02774724	1.52	0.67523687
0.02989830	1.54	0.69607779
0.03212758	1.56	0.71667346
0.03443464	1.58	0.73703423
0.03681896	1.60	0.75717019
0.03928007	1.62	0.77709104
0.04181743	1.64	0.79680620
0.04443050	1.66	0.81632488
0.04711874	1.68	0.83565595
0.04988157	1.70	0.85480811
0.05271841	1.72	0.87378982

Table 2 (cont'd)

For $n = \frac{3}{4}$

Z^*	U^*	δ^*
0.05562866	1.74	0.89260934
0.05861171	1.76	0.91127484
0.06166955	1.78	0.92979424
0.06479372	1.80	0.94817539
0.06799137	1.82	0.96642596
0.07125925	1.84	0.98455361
0.07411692	1.857	0.99999505

Table 2 (cont'd)

For $n = \frac{1}{2}$

Z^*	U^*	δ^*
0.00000000	1.00	0.00000000
0.00023198	1.02	0.03952822
0.00069818	1.04	0.07814439
0.00133372	1.06	0.11589381
0.00212294	1.08	0.15281887
0.00305833	1.10	0.18895930
0.00413561	1.12	0.22435251
0.00535207	1.14	0.25903370
0.00670584	1.16	0.29303606
0.00819552	1.18	0.32639093
0.00982001	1.20	0.35912788
0.01157836	1.22	0.39127491
0.01346974	1.24	0.42285863
0.01549338	1.26	0.45390417
0.01764852	1.28	0.48443550
0.01993442	1.30	0.51447540
0.02235035	1.32	0.54404554
0.02489557	1.34	0.57316661
0.02756931	1.36	0.60185841
0.03037080	1.38	0.63013981

Table 2 (cont'd)

For $n = \frac{1}{2}$

Z^*	U^*	δ^*
0.03329926	1.40	0.65802891
0.03635386	1.42	0.68554310
0.03953379	1.44	0.71269902
0.04283818	1.46	0.73951272
0.04626618	1.48	0.76599966
0.04981689	1.50	0.79217476
0.05348939	1.52	0.81805238
0.05728277	1.54	0.84364651
0.06119606	1.56	0.86897064
0.06522830	1.58	0.89403788
0.06937849	1.60	0.91886104
0.07364562	1.62	0.94345257
0.07802865	1.64	0.96782462
0.08252651	1.66	0.99198912
0.08406192	1.667	1.00003453

Table 2 (cont'd)

For $n = \frac{1}{4}$

Z^*	U^*	δ^*
0.00000000	1.00	0.00000000
0.00096809	1.02	0.05892007
0.00251835	1.04	0.11658444
0.00438575	1.06	0.17305892
0.00652720	1.08	0.22840543
0.00892183	1.10	0.28268220
0.01155723	1.12	0.33594422
0.01442506	1.14	0.33824341
0.01751921	1.16	0.43962898
0.02083492	1.18	0.49014757
0.02436827	1.20	0.53984354
0.02811589	1.22	0.58875903
0.03207479	1.24	0.63693437
0.03624224	1.26	0.68440798
0.04061565	1.28	0.73121670
0.04519261	1.30	0.77739594
0.04997072	1.32	0.82297970
0.05494768	1.34	0.86800076
0.06012119	1.36	0.91249081
0.06548896	1.38	0.95648052
0.07104870	1.40	1.00000077

Table 3

Parameters $\beta^{1/2}$ and $\frac{X}{(Pr/9)^{1/3}}$ as Function of Z^*

For Newtonian Fluid

Z^*	$\beta^{1/2}$	$\frac{X}{(Pr/9)^{1/3}}$
0.000005	12.170	341.000
0.000006	11.750	304.000
0.000008	11.000	255.000
0.000010	10.410	219.000
0.000020	9.050	140.000
0.000040	7.880	110.000
0.000060	7.280	90.800
0.000080	7.000	80.500
0.000100	6.720	72.600
0.000200	5.850	52.100
0.000400	5.100	37.500
0.000600	4.770	31.400
0.000800	4.530	27.600
0.001000	4.360	25.000
0.002000	3.870	18.400
0.004000	3.430	13.500
0.006000	3.220	11.300
0.008000	3.030	9.830
0.010000	2.970	9.040

Table 3 (cont'd)

Z^*	$\beta^{1/2}$	$\frac{X}{(Pr/9)^{1/3}}$
0.020000	2.650	6.660
0.040000	2.390	4.930
0.060000	2.280	4.200
0.080000	2.190	3.720
0.100000	2.140	3.410

Table 4

Parameters $\beta^{1/2}$ and $\frac{X}{(Pr/9)^{1/3}}$ as Function of Z^*

For Non-Newtonian Fluid

For n = 3/4

Z^*	$\beta^{1/2}$	$\frac{X}{(Pr/9)^{1/3}}$
0.0001	7.400	67.822
0.0002	6.150	45.858
0.0004	5.200	32.030
0.0006	4.660	25.750
0.0008	4.350	22.281
0.0010	4.100	19.814
0.0020	3.530	14.261
0.0040	3.070	10.349
0.0060	2.840	8.605
0.0080	2.740	7.682
0.0100	2.620	6.917
0.0200	2.479	5.859
0.0300	2.262	4.408
0.0400	2.195	3.943
0.0500	2.148	3.620
0.0600	2.111	3.375
0.0700	2.084	3.185
0.0741	2.076	3.085

Table 4 (cont'd)

For n = 1/4

Z^*	$\beta^{1/2}$	$\frac{X}{(Pr/9)^{1/3}}$
0.0001	22.000	106.900
0.0002	15.500	69.949
0.0004	11.700	48.508
0.0006	10.500	41.143
0.0008	9.750	36.539
0.0010	9.150	33.041
0.0020	7.100	22.662
0.0040	5.700	15.952
0.0060	4.950	12.779
0.0080	4.511	10.981
0.0100	4.228	9.825
0.0200	3.533	7.069
0.0300	3.142	5.740
0.0400	2.985	5.040
0.0500	2.812	4.562
0.0600	2.758	3.595
0.0700	2.632	3.082
0.0710	2.627	3.063

Table 4 (cont'd)

For n = 1/2

Z^*	$\beta^{1/2}$	$\frac{X}{(Pr/9)^{1/3}}$
0.0001	11.200	78.289
0.0002	9.300	56.943
0.0004	7.500	39.832
0.0006	6.600	32.145
0.0008	6.080	27.877
0.0010	5.710	24.915
0.0020	4.680	17.480
0.0040	3.900	12.405
0.0060	3.540	10.222
0.0080	3.320	8.940
0.0100	3.150	8.031
0.0200	2.751	5.895
0.0300	2.570	4.959
0.0400	2.456	4.393
0.0500	2.380	4.011
0.600	2.326	3.731
0.0700	2.283	3.511
0.0800	2.248	3.331
0.0841	2.236	3.268

Table 5

The Local Nusselt Number (Nu_z) as Function of Dimensionless Distance (Z^*) and Prandtl Number (Pr) For Newtonian Fluid

Z^* \ Pr	5	10	15	20	40	100
0.000005	312.982	395.843	451.836	497.752	626.521	850.483
0.000006	279.397	352.176	402.563	442.885	557.100	758.660
0.000008	233.471	296.188	337.627	372.335	467.510	636.602
0.000010	206.151	259.238	296.187	327.545	411.528	559.342
0.000020	142.431	179.591	205.696	226.763	284.983	386.997
0.000040	100.788	127.247	145.793	160.471	202.313	274.752
0.000060	[illegible]	104.743	118.891	132.120	166.605	226.275
0.000080	73.572	92.798	106.354	117.076	147.636	200.552
0.000100	66.293	83.630	95.850	105.563	133.087	180.817
0.000200	47.363	59.862	68.638	75.571	95.354	129.599
0.000400	33.931	42.934	49.257	54.240	68.472	93.125
0.000600	28.333	35.851	41.139	45.329	57.246	77.887
0.000800	24.858	31.453	36.091	39.773	50.241	68.393
0.001000	22.502	28.439	32.642	35.970	45.460	61.905
0.002000	16.427	20.829	23.851	26.438	33.263	45.463
0.004000	11.899	15.122	17.361	19.159	24.191	33.142
0.006000	9.847	12.570	14.440	15.943	20.211	27.665
0.008000	8.484	10.846	12.511	13.812	17.526	23.963
0.010000	7.770	9.925	11.447	12.635	16.075	21.942

Table 5 (cont'd)

Z^* \ Pr	5	10	15	20	40	100
0.020000	5.597	7.175	8.295	9.167	11.724	16.077
0.040000	4.063	5.194	6.001	6.663	8.555	11.758
0.060000	3.970	4.553	5.036	5.596	7.207	9.940
0.080000	3.891	3.787	4.405	4.880	6.308	8.727
0.100000	2.622	3.410	3.988	4.441	5.740	7.966

Table 6

The Local Nusselt Number (Nu_z) as Function of Dimensionless Distance (Z^*) and Prandtl Number (Pr) For Non-Newtonian Fluid

For $n = 3/4$

Z^* \ Pr	5	20	40	100	150	200
0.0001	61.866	98.545	124.294	168.878	193.432	212.965
0.0002	41.645	66.451	83.857	114.004	130.607	143.810
0.0004	28.919	46.244	58.399	79.458	91.048	100.273
0.0006	23.143	37.065	46.841	63.765	73.088	80.501
0.0008	19.950	31.997	40.453	55.097	63.166	69.586
0.0010	17.678	28.389	35.910	48.935	56.107	61.817
0.0020	12.569	20.279	25.689	35.061	40.299	44.331
0.0040	8.871	14.566	18.492	25.291	29.040	32.018
0.0060	7.406	12.016	15.282	20.936	24.049	26.527
0.0080	6.518	10.669	13.587	18.634	21.414	23.620
0.0100	5.814	9.552	12.174	16.723	19.226	21.214
0.0200	4.235	7.038	9.005	12.421	14.301	15.796
0.0300	3.514	5.891	7.559	10.457	12.052	13.321
0.0400	3.088	5.212	6.705	9.295	10.722	11.856
0.0500	2.792	4.740	6.112	8.489	9.787	10.838
0.0600	2.569	4.384	5.662	7.880	9.010	10.069
0.0700	2.396	4.108	5.315	7.409	8.561	9.475
0.0741	2.315	4.015	5.145	7.265	8.405	9.195

Table 6 (cont'd)

For n = 1/2

Z^* \ Pr	5	20	40	100	150	200
0.0001	71.501	113.846	143.564	195.020	223.375	245.907
0.0002	51.849	82.648	104.264	141.691	162.309	178.699
0.0004	36.100	57.644	72.764	98.944	113.377	124.838
0.0006	29.026	46.411	58.615	79.744	91.385	100.634
0.0008	25.098	40.172	50.756	69.081	79.181	87.203
0.0010	22.374	35.845	45.303	61.684	70.703	77.874
0.0020	15.529	24.984	31.613	43.105	49.433	54.480
0.0040	10.860	17.568	22.276	30.428	34.920	38.493
0.0060	8.854	14.379	18.255	24.973	28.703	31.614
0.0080	7.675	12.507	15.899	21.774	25.009	27.583
0.0100	6.840	11.178	14.225	19.501	22.408	24.709
0.0200	4.877	8.059	10.296	14.168	16.302	18.001
0.0300	4.019	6.696	8.575	11.833	13.629	15.055
0.0400	3.499	5.869	7.533	10.418	12.007	13.272
0.0500	3.149	5.310	6.829	9.462	10.915	12.069
0.0600	2.895	4.903	6.317	8.766	10.116	11.191
0.0700	2.693	4.582	5.911	8.207	9.486	10.497
0.0800	2.529	4.321	5.581	7.767	8.972	9.931
0.0841	2.471	4.228	5.465	7.610	8.792	9.732

Table 6 (cont'd)

For n = 1/4

Z^* \ Pr	5	20	40	100	150	200
0.0001	97.838	155.653	196.225	266.502	305.223	335.974
0.0002	63.826	101.657	128.266	174.195	199.520	219.645
0.0004	44.083	70.319	88.737	120.615	138.181	152.149
0.0006	37.306	59.558	75.249	102.220	117.118	128.958
0.0008	33.071	52.829	66.701	90.718	103.950	114.470
0.0010	29.851	47.477	60.261	81.984	93.951	103.459
0.0020	20.296	32.554	41.156	55.483	64.258	70.782
0.0040	14.120	22.750	28.807	39.288	45.064	49.660
0.0060	11.205	18.115	22.965	31.363	35.986	39.668
0.0080	9.551	15.487	19.657	26.870	30.847	34.007
0.0100	8.488	13.799	17.528	23.983	27.541	30.365
0.0200	5.955	9.773	12.455	17.099	19.660	21.694
0.0300	4.735	7.834	10.012	13.781	15.861	17.511
0.0400	4.093	6.812	8.723	12.034	13.861	15.310
0.0500	3.654	6.114	7.844	10.839	12.490	13.806
0.0600	3.188	5.601	7.196	9.961	11.484	12.697
0.0700	2.911	5.203	6.695	9.280	10.704	11.837
0.0711	2.889	5.171	6.654	9.225	10.642	11.769

Table 7

The Average Nusselt Number (Nu_{avg}) as Function of Dimensionless Distance (Z^*) and Prandtl Number (Pr)

For Newtonian Fluid

Z^* \ Pr	5	10	15	20	40	100
0.000005	500.870	630.085	721.262	793.861	1000.205	1357.481
0.000006	465.832	587.085	672.001	739.210	931.837	1264.566
0.000008	413.121	520:551	596.002	655.419	826.372	1119.675
0.000010	374.307	430.845	539.801	593.936	748.311	1015.147
0.0000[illegible]	268.654	320.405	389.906	428.970	539.158	732.577
0.0000[illegible]0	194.825	237.702	282.951	311.485	390.589	533.799
0.000060	163.550	200.801	237.974	261.323	327.721	447.523
0.000080	144.462	178.106	209.987	230.494	289.795	394.391
0.000100	130.775	161.685	189.984	208.397	262.436	357.515
0.000200	92.835	117.347	136.494	150.203	189.221	258.765
0.000400	69.192	84.170	97.253	106.604	135.614	183.384
0.000600	56.462	69.453	80.001	87.732	111.410	151.255
0.000800	49.034	60.596	69.756	76.601	97.052	132.197
0.001000	43.987	54.577	62.704	68.988	87.243	118.956
0.002000	31.044	39.283	50.356	49.693	62.626	84.737
0.004000	22.322	28.145	33.971	35.841	45.213	61.362
0.000000	18.488	23.368	28.566	29.665	37.475	50.991
0.008000	16.086	20.452	24.763	25.906	32.786	44.743

Table 7 (cont'd)

Z^* \ Pr	5	10	15	20	40	100
0.010000	14.467	18.421	22.210	23.342	29.549	40.408
0.020000	10.359	13.269	15.854	16.976	21.277	29.206
0.040000	7.505	9.587	11.304	12.183	15.545	21.351
0.060000	6.453	7.973	9.353	10.126	12.961	17.801
0.080000	5.402	6.990	8.196	8.903	11.425	15.669
0.100000	4.872	6.312	7.394	8.054	10.332	14.194

Table 8

The Average Nusselt Number (Nu_{avg}) as Function of Dimensionless Distance (Z^*) and Prandtl Number (Pr) For Non-Newtonian Fluid

For n = 3/4

Z^* \ Pr	5	20	40	100	150	200
0.0001	159.815	237.225	301.682	450.700	493.117	559.700
0.0002	103.527	158.107	199.055	290.854	324.555	364.855
0.0004	68.654	106.054	133.281	191.939	215.280	240.437
0.0006	54.331	84.362	105.857	151.281	170.188	189.625
0.0008	46.075	71.526	90.011	127.962	144.393	160.468
0.0010	40.548	63.029	79.418	112.574	127.117	141.372
0.0020	27.515	43.462	54.866	76.285	85.905	95.435
0.0040	18.898	30.194	38.052	52.393	59.703	66.467
0.0060	15.275	24.443	30.888	42.528	47.768	53.912
0.0080	13.181	21.144	26.761	36.821	41.476	46.659
0.0100	11.769	18.920	23.979	32.957	37.221	41.787
0.0200	8.327	13.466	17.125	23.511	26.651	29.764
0.0300	6.928	11.114	14.126	19.446	22.099	24.649
0.0400	5.940	9.715	12.386	17.042	19.399	21.611
0.0500	5.342	8.764	11.185	15.406	17.564	19.544
0.0600	4.896	8.062	10.300	14.202	16.209	18.025
0.0700	4.559	7.513	9.610	13.261	15.152	16.844
0.0741	4.429	7.320	9.368	12.932	14.780	16.539

Table 8 (cont'd)

For n = 1/2

Z^* \ Pr	5	20	40	100	150	200
0.0001	203.591	313.543	407.047	552.605	632.807	696.703
0.0002	131.795	203.305	265.029	356.307	407.651	448.354
0.0004	86.947	135.251	174.761	235.658	269.083	296.775
0.0006	68.693	107.173	137.907	186.402	212.851	234.533
0.0008	58.224	91.125	116.955	158.175	180.737	199.202
0.0010	51.219	80.502	103.084	139.583	159.458	175.788
0.0020	34.361	54.605	69.442	95.993	107.825	119.091
0.0040	23.530	37.552	47.621	65.595	74.013	81.645
0.0060	18.926	30.275	38.414	52.897	59.808	65.432
0.0080	16.227	26.025	33.035	45.447	51.431	56.722
0.0100	14.422	23.160	29.408	40.438	45.825	50.538
0.0200	10.026	16.231	20.604	28.407	32.465	35.575
0.0300	8.151	13.245	16.856	23.228	26.572	29.168
0.0400	7.052	11.496	14.643	20.181	23.126	25.403
0.0500	6.300	10.312	13.147	18.128	20.781	22.848
0.0600	5.749	9.439	12.048	16.616	19.062	20.975
0.0700	5.323	8.766	11.203	15.448	17.731	19.524
0.0800	4.881	8.224	10.518	14.511	16.664	18.357
0.0841	4.861	8.033	10.276	14.181	16.286	17.941

Table 8 (cont'd)

For n = 1/4

Z^* \ Pr	5	20	40	100	150	200
0.0001	478.500	763.800	959.400	1299.650	1481.865	1656.232
0.0002	275.515	439.475	554.513	751.881	859.903	958.107
0.0004	162.754	260.702	328.258	445.932	508.952	566.555
0.0006	122.002	195.325	246.010	[illegible]	380.967	440.367
0.0008	100.125	160.225	202.407	274.221	312.975	360.275
0.0010	[illegible]	138.040	174.115	236.367	269.980	309.629
0.0020	55.373	87.776	111.302	150.680	172.491	194.813
0.0040	35.837	57.132	72.258	97.591	111.370	126.405
0.0060	27.993	44.673	56.567	76.563	87.247	98.603
0.0080	23.560	37.681	47.675	56.586	73.685	83.078
0.0100	20.642	33.064	41.840	38.093	64.748	72.933
0.0200	13.781	22.195	28.095	30.429	43.699	48.736
0.0300	10.932	17.655	22.396	26.009	34.899	38.873
0.0400	9.304	15.060	19.123	23.082	29.862	33.218
0.0500	8.206	13.348	16.948	20.956	26.500	29.480
0.0600	7.399	12.085	15.373	19.327	24.058	26.783
0.0700	6.776	11.123	14.166	18.177	22.193	24.700
0.0711	6.719	11.035	14.055	18.068	22.021	24.510

Table 9-a

The Constants "a" and "b" of $Nu_z = bZ^{*a}$

as Determined from Figure 4

For n = 1/4

Pr	a (For $Z^* > 0.0015$)	b (For $Z^* > 0.0015$)	a (For $Z^* < 0.0015$)	b (For $Z^* < 0.0015$)
5	-0.527	0.781	-0.571	1.490
20	-0.527	1.213	-0.571	2.396
40	-0.527	1.571	-0.571	3.038
100	-0.527	2.153	-0.571	4.118
150	-0.527	2.435	-0.571	4.712
200	-0.527	2.712	-0.571	5.228

For n = 1/2

Pr	a (For $Z^* > 0.0015$)	b (For $Z^* > 0.0015$)	a (For $Z^* < 0.0015$)	b (For $Z^* < 0.0015$)
5	-0.514	0.848	-0.475	0.788
20	-0.514	1.322	-0.475	1.288
40	-0.514	1.704	-0.475	1.639
100	-0.514	2.302	-0.475	2.241
150	-0.514	2.630	-0.475	2.558
200	-0.514	2.897	-0.475	2.823

Table 9-a (cont'd)

For n = 3/4

Pr	a	b	a	b
5	-0.519	0.492	-0.445	0.757
20	-0.519	0.798	-0.445	1.040
40	-0.519	0.994	-0.445	1.588
100	-0.519	1.373	-0.445	2.183
150	-0.519	1.569	-0.445	2.508
200	-0.519	1.724	-0.445	2.760

Table 9-b

The Constants "a" and "b" of $Nu_{avg} = bZ^{*a}$

as Determined from Figure 5

For n = 1/4

Pr	a ($Z^* > 0.0015$)	b ($Z^* > 0.0015$)	a ($Z^* < 0.0015$)	b ($Z^* < 0.0015$)
5	-0.722	0.582	-0.571	1.490
20	-0.722	0.931	-0.571	2.396
40	-0.722	1.174	-0.571	3.038
100	-0.722	1.593	-0.571	4.118
150	-0.722	1.819	-0.571	4.712
200	-0.722	2.077	-0.571	5.288

For n = 1/2

Pr	a ($Z^* > 0.0015$)	b ($Z^* > 0.0015$)	a ($Z^* < 0.0015$)	b ($Z^* < 0.0015$)
5	-0.593	0.848	-0.513	1.363
20	-0.593	1.322	-0.513	2.214
40	-0.593	1.704	-0.513	2.815
100	-0.593	2.302	-0.513	3.880
150	-0.593	2.630	-0.513	4.397
200	-0.593	2.897	-0.513	4.835

Table 9-b (cont'd)

For n = 3/4

Pr	a	b	a	b
5	-0.584	0.717	-0.500	1.187
20	-0.584	1.109	-0.500	1.918
40	-0.584	1.359	-0.500	2.432
100	-0.584	1.998	-0.500	3.355
150	-0.584	2.247	-0.500	3.790
200	-0.584	2.506	-0.500	4.242

VITA AUCTORIS

1940 Born in Sault Ste. Marie, Ontario, on July 31, 1940.

1953 Completed elementary school at St. Stanislaus Separate School, Sault Ste. Marie, in June.

1958 Obtained senior matriculation on graduation from Sault Ste. Marie Collegiate Institute.

1958 Awarded Sir James Dunn Scholarship on entrance to Assumption University of Windsor.

1962 Received the Degree of Bachelor of Applied Science in Chemical Engineering from Assumption University of Windsor, Windsor, Ontario.

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